Performance of Asynchronous Two-Relay Two-Hop Wireless Cooperative Networks

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Abstract

In wireless cooperative networks, the asynchronism between the relays can be a source of diversity which is similar in its essence to the multipath diversity of frequency selective channels. In this context, an asynchronous two-relay cooperative wireless network is studied for Decode-and-Forward (DF) and Amplify-and-Forward (AF) protocols. The outage probability in the high Signal to Noise Ratio (SNR) regime is derived and the impact of the relative delay between the two relays on this outage probability is evaluated. It is shown that for a sufficiently high relative delay, the outage probability performance becomes independent from the relative delay and approaches from the synchronous protocol performance. Besides, an optimization of the power distribution between the transmitting nodes of the network is carried out in the high SNR regime based on the outage probability minimization. Moreover, the Diversity Multiplexing Tradeoff (DMT) of the network is characterized for the two cooperative protocols DF and AF. The DMT curve does not depend on the relative delay as long as the latter is non-zero.

Keywords: *Asynchronous Relay Channel, Cooperative Diversity, Cooperative Wireless Networks, Diversity-Multiplexing Tradeoff, Outage Probability.*

I. INTRODUCTION

In wireless communications, the signal transmission is hampered by several attenuation factors such as channel fading, shadowing, and path loss. It is well known that Multiple Input Multiple Output (MIMO) techniques improve the reliability and/or the data rate. However, in some devices, it is not possible to implement many antennas due to size, hardware limitations, or cost constraints. Therefore, the idea of cooperation between the nodes of a network, in a manner to create a virtual MIMO system, has attracted attention as an efficient way to allow these single-antenna devices to benefit of spatial diversity [1] [2]. Many cooperation protocols exist, and three main protocol classes are the Amplify-and-Forward (AF), the Decode-and-Forward (DF) and the Compress-and-Forward (CF) protocols. In general, a cooperation protocol consists in two phases. In the first one, the source broadcasts its message in the network. During the second phase, the relay nodes retransmit the source information to the destination, after a special processing that depends upon the cooperation protocol.

So far, most existing research on cooperative diversity assumes perfect synchronization among cooperative nodes [2]–[6]. Under this assumption, distributed space-time-coded cooperative diversity approach achieves diversity gains in the order of the number of available transmitting nodes in a relay network. However, this assumption is difficult to satisfy in distributed networks like ad-hoc and sensor networks [7]. The lack of perfect synchronization between relay nodes has been recently considered in some works. Space-Time codes and Space-Frequency codes were used in [8]–[11] to combat the imperfect synchronization and to tolerate the delay asynchronism in the cooperative networks.

In other contributions, asynchronism is considered as beneficial: when the relays transmit delayed replicas of a given message towards the destination, a diversity effect similar to multipath diversity comes into play. In this line of thought, the author of [7] derived the Diversity Multiplexing Tradeoff (DMT) curve of two asynchronous cooperative schemes using the DF protocol and showed that they achieve the same performance as the synchronous Space-Time coded scheme. In [12], the authors considered two different models of asynchronous cooperative relay networks and propose a variant of the Slotted Amplify and Forward (SAF) protocol which asymptotically achieves the transmit diversity bound for both these models.

In this paper, we study the outage probability of an asynchronous two-relay two-hop wireless network with single-antenna nodes for the DF and the AF protocols, in the absence of a direct link between source and destination. The exact evaluation of the outage probability P_o for any value of the Signal to Noise Ratio (SNR) ρ is known to be a difficult task. A simpler problem is to study the behavior of P_o at high values of ρ. Usually, there exists an integer $d > 0$ for which $\rho^d P_o$ converges as $\rho \to \infty$ to a positive value, in which case d is precisely the diversity gain of the relaying scheme. The constant $\xi = \lim_{\rho\to\infty} \rho^d P_o$ provides very useful information on the behavior of P_o. For instance, it can be used to compare the performance of two protocols having the same diversity gain or to optimize the power distribution among the transmitting nodes. This has been done in [13] and in [14], two papers devoted to synchronous protocols. In [13], ξ is called "outage gain". In this paper, we shall extend the definition of the outage gain as follows: assume there exists a real function $f > 0$ such that $\lim_{\rho\to\infty} f(\rho)P_o = \xi > 0$, then ξ is called the outage gain associated with f. In most of the paper, f will take the form $f(\rho) = \rho^2$ (there exists a diversity gain of 2) except for one case with the AF protocol in Section IV-A3.

In this work we derive the outage gain for the different asynchronous scenarios introduced above, and we minimize it with respect to the power distribution. Our simulation results show in particular that the asynchronous DF protocol using the same codebook for the two relays has approximately the same outage probability performance as the DF protocol requiring accurate time synchronization with two different codebooks employed by the relays. For the asynchronous AF protocol, we remarked that in the most general case, the outage gain is very difficult to evaluate. However, we give closed form expressions and/or simulations for the outage gain in three pertinent situations which we believe are interesting to study.

The outage gain is a pertinent performance criterion in the frequent situations where data rate is fixed independently of the SNR. In the case where the rate is assumed to grow with the SNR, the widely used performance measure is the DMT. Complementing [7] in which an asynchronous two-relay network using DF protocol with a direct source-destination link was considered, we also evaluate the DMT of the considered network for both DF and AF cooperative protocols. Contrary to [7], our DMT derivation is valid for any non zero relative delay between the two relays.

The contributions of this paper can be summarized as follows:

- A general formalism to calculate the outage gain for asynchronous relay networks using Decode-and-Forward or Amplify-and-Forward cooperative protocol, which have not been done so far to the best of our knowledge.
- A power allocation optimization based on the minimization of the outage gain factor for asynchronous relay networks.
- A derivation of the DMT for both asynchronous DF and AF protocols with no particular assumption on the transmission model (all channels are random fading channels) and for any delay profile (non-zero relative delay between the two relays).

The paper is organized as follows. In Section II, the system model is described. In Sections III and IV, the outage gain is derived for the DF and AF protocols respectively. For the DF protocol, the outage gain is derived for a general class of radio channels, but in the AF protocol case, it is calculated in some particular configurations of the network channels because of the mathematical complexity in the general case. Section V describes the power optimization method. The DMT expression of the network is given in Section VI. Numerical results are provided in Section VII. Finally, some conclusions are drawn in Section VIII.

II. System Model, Assumptions and Notations

We consider a wireless network that consists in a source S, two relay nodes R_1 and R_2, and a destination D as shown in Figure 1. All nodes have a single antenna. The relays operate in the half-duplex mode, which prohibits them from transmitting and receiving at the same time [3]. We assume that there is no direct link between the source and the destination because the channel gain between these two nodes is practically zero due for instance to the large distance separating these two nodes.

In this work, we consider orthogonal cooperative protocols which are generally divided into two phases, each with duration $T/2$ channel uses. During Phase I, the source broadcasts its message towards the relays. During Phase II, the source stops transmitting and the active relays forward the source information to the destination. For the DF protocol, only the relays that succeed in decoding the source message transmit in the second phase. Denote by $\mathcal{D}(s)$ the set of relays that succeed in decoding the source message. The cardinality of this set satisfies $|\mathcal{D}(s)| \in \{0, 1, 2\}$. When the two relays succeed in decoding the source message ($|\mathcal{D}(s)| = 2$), it is assumed for implementation simplicity reasons as in [7] that the relays employ the same Gaussian codebook to transmit in Phase II. For the AF protocol, the two relays transmit in Phase II the amplified versions of the message broadcasted by the source in Phase I.

Due to the distributed nature of the network, a different time delay is introduced on each relay-destination path; we denote by τ_1 and τ_2 the delays of the signals received by the destination from relays R_1 and R_2 respectively.

In this paper, the parameter ρ will represent a certain power budget of the network and so it will be proportional to the transmitted powers of the network nodes (S, R_1 and R_2), or equivalently ρ will be proportional to the Signal to Noise Ratio (SNR).

Let us assume that the source S transmits the Gaussian codeword $(x(k))_{k=1}^{T/2} \in \mathbb{C}^{T/2}$ during Phase I. Denote by $\beta_0\rho$ the power of the source where $\beta_0 \in (0, 1)$ is a power coefficient given to the source. The amplitude gain applied to $x(k)$ is $\sqrt{\beta_0\rho}$. The signal $y_i(k)$ received by relay R_i ($i = 1, 2$) at the output of this relay's matched filter is

$$y_i\,(k) = \sqrt{\beta_0\rho}\, H_{0i}\, x(k) + n_i(k), \quad i = 1, 2.$$

Assuming channels are frequency flat fading channels, the complex random variable H_{0i} is the channel gain between the source S and the relay R_i, and n_i is the Additive White Gaussian Noise (AWGN) at R_i.

For the DF protocol, if the relay R_i succeeds to decode $x(k)$, this relay transmits the same codeword in Phase II. The continuous time signal y_{DF} received by the destination D during

Phase II is:

$$y_{DF}\left(t\right)=\sum_{i:R_i\in\mathcal{D}(s)}\sqrt{\beta_i\rho}\,H_{i3}\,\sum_k x(k)\Phi(t-\tau_i-k)\;+\;n_3(t) \tag{1}$$

where $\beta_i\rho$ is the power of relay R_i, H_{i3} is the channel between relay R_i and D, $n_3(t)$ is the AWGN at D, and $\Phi(t)$ is the equivalent transmitter receiver filter.

In the AF case, the relay R_i simply transmits the signal $\sqrt{A_i}y_i(k)$ towards the destination where A_i is the power gain applied by R_i. The signal received by the destination D is

$$\begin{aligned} y_{AF}(t) &= \sum_{i=1,2}\sqrt{A_i}H_{i3}\sum_k y_i(k)\Phi(t-\tau_i-k)\;+\;n_3(t)\\ &= \sum_{i=1,2}\left(\sqrt{A_i}H_{i3}\sum_k\left(\sqrt{\beta_0\rho}H_{0i}x(k)+n_i(k)\right)\Phi(t-\tau_i-k)\right)\;+\;n_3(t). \end{aligned} \tag{2}$$

Our assumptions and notations are the following. The complex channel gains H_{ij}, available only at the receivers, are assumed to be independent random variables. We assume that the joint density $f_{H_{ij}}(x,y)$ of $(\mathrm{Re}(H_{ij}),\mathrm{Im}(H_{ij}))$ is continous and positive at $(0,0)$, and we denote by $b_{ij}=f_{H_{ij}}(0,0)$ its value at zero. All AWGN at relays and at destination are independent with unit variance. The delays τ_i are also known to the destination. Without loss of generality, we consider that $\tau_2\geq\tau_1$ so that the relative delay between the two relays is $\Delta=\tau_2-\tau_1\geq 0$. Finally, the filter $\Phi(t)$ is assumed to be a perfect low pass filter with a transfer function $\mathbf{1}_{[-1/2,1/2]}(f)$ where $\mathbf{1}_{\mathcal{A}}$ is the indicator function of set $\mathcal{A}$.

The outage gain derivations below often make use of the power channel gains $G_{ij}=|H_{ij}|^2$. We shall denote by $f_{G_{ij}}$ the density of G_{ij}. The following lemma will be useful:

Lemma 1: Assuming $f_{H_{ij}}$ is continuous at $(0,0)$, the density $f_{G_{ij}}$ is right continuous at zero, and $c_{ij}=f_{G_{ij}}(0^+)$ satisfies $c_{ij}=\pi b_{ij}$.

This lemma is proven in Appendix A. As an example, assume the H_{ij} are complex Gaussian circular with $\mathbb{E}H_{ij}=0$, and $\mathbb{E}|H_{ij}|^2=\sigma_{ij}^2$ (Rayleigh channels). In this case, one can check that $b_{ij}=1/(\pi\sigma_{ij}^2)$ and $c_{ij}=1/\sigma_{ij}^2$.

III. Outage gain of the DF Protocol

The outage probability of our protocol is $P_o=\mathbb{P}\left[\mathcal{I}\leq R\right]$ where $\mathcal{I}$ is the mutual information between the source and the destination (to be specified below) and R is the source transmission rate in nats per channel use. We shall show that the outage gain $\xi_{DF} > 0$ of this protocol is associated with the function $f(\rho) = \rho^2$, *i.e.*, our protocol has a diversity gain of 2. To show that $\rho^2 P_o$ converges and to derive the expression of ξ_{DF}, we divide the outage probability into three components:

$$\begin{aligned} P_o &= \mathbb{P}\left[|\mathcal{D}(s)| = 0\right] \ + \ \mathbb{P}\left[|\mathcal{D}(s)| = 1\right] \, \mathbb{P}\left[\mathcal{I} \leq R \ / \ \ |\mathcal{D}(s)| = 1\right] \\ &\quad + \ \mathbb{P}\left[|\mathcal{D}(s)| = 2\right] \, \mathbb{P}\left[\mathcal{I} \leq R \ / \ \ |\mathcal{D}(s)| = 2\right] \\ &= P_{o,0} + P_{o,1} + P_{o,2}. \end{aligned} \tag{3}$$

Based on this decomposition, we obtain the following proposition which is the main result of this section:

Proposition 1: Consider the asynchronous two-relay two-hop network and a DF protocol as described in Section II above. For a given rate R*, the outage probability* P_o *satisfies* $\rho^2 P_o \underset{\rho \to \infty}{\longrightarrow} \xi_{DF} = \xi_0 + \xi_1 + \xi_2$ *where*

$$\xi_0 = \frac{c_{01} c_{02}}{\beta_0^2} \left(e^{2R} - 1\right)^2 \tag{4}$$

$$\xi_1 = \left(\frac{c_{13} c_{02}}{\beta_1 \beta_0} + \frac{c_{23} c_{01}}{\beta_2 \beta_0}\right) \left(e^{2R} - 1\right)^2 \tag{5}$$

$$\xi_2 = \frac{c_{13} c_{23}}{\pi^2 \beta_1 \beta_2} \int_{\{(z_1,z_2) \in \mathbb{C}^2, \, \mathcal{I}(z_1,z_2,\Delta) \leq R\}} dz_1 \, dz_2 \tag{6}$$

with $\mathcal{I}(z_1, z_2, \Delta) = \dfrac{1}{2} \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + |z_1 + z_2 \exp\left(-2\imath\pi f \Delta\right)|^2\right) df$*. Here it is understood that* $dz_i = dx_i \, dy_i$ *where* $z_i = x_i + \imath y_i$ *with* $\imath^2 = -1$*. For positive integer values of* Δ*,* ξ_2 *is given by*

$$\xi_2 = \frac{c_{13} c_{23}}{\beta_1 \beta_2} \, e^{2R} \left(e^{2R} - 2R - 1\right) \ . \tag{7}$$

More generally, ξ_2 *satisfies*

$$\frac{c_{13} c_{23}}{\beta_1 \beta_2} e^{2R \frac{\Delta}{\lceil \Delta \rceil}} \left(e^{2R \frac{\Delta}{\lceil \Delta \rceil}} - 2R \frac{\Delta}{\lceil \Delta \rceil} - 1\right) \leq \xi_2 \leq \frac{c_{13} c_{23}}{\beta_1 \beta_2} e^{2R \frac{\Delta}{\lfloor \Delta \rfloor}} \left(e^{2R \frac{\Delta}{\lfloor \Delta \rfloor}} - 2R \frac{\Delta}{\lfloor \Delta \rfloor} - 1\right) \tag{8}$$

where $\lceil \Delta \rceil$ *(resp.* $\lfloor \Delta \rfloor$*) is the smallest integer* $\geq \Delta$ *(resp. largest integer* $\leq \Delta$*) and* $\Delta \geq 1$ *for the upper bound.*

The parameters c_{01}*,* c_{02}*,* c_{13} *and* c_{23} *in Equations* (4)-(8) *represent the values at zero of the densities* $f_{G_{01}}$*,* $f_{G_{02}}$*,* $f_{G_{13}}$ *and* $f_{G_{23}}$ *respectively.*

Proof: The proof will show that $\xi_l = \lim_{\rho \to \infty} \rho^2 P_{o,l}$ for $l = 0, 1, 2$. We shall treat in turn $P_{o,0}$, $P_{o,1}$ and $P_{o,2}$.

Let us begin with $P_{o,0}$. We have

$$\rho^2 P_{o,0} = \rho^2 \prod_{i=1}^{2} \mathbb{P}\left[\frac{1}{2}\log\left(1+\rho\beta_0 G_{0i}\right) \le R\right] = \rho^2 \prod_{i=1}^{2} \int_{\frac{1}{2}\log(1+\rho\beta_0 x_i)\le R} f_{G_{0i}}\left(x_i\right) dx_i .$$

By making the variable changes $u_i = \rho\beta_0 x_i$, we obtain

$$\rho^2 P_{o,0} = \frac{1}{\beta_0^2} \prod_{i=1}^{2} \int_{\frac{1}{2}\log(1+u_i)\le R} f_{G_{0i}}\left(\frac{u_i}{\rho\beta_0}\right) du_i .$$

We now let $\rho \to \infty$. By applying Lebesgue's Dominated Convergence Theorem and by relying on the right continuity of the $f_{G_{0i}}$ at zero [13] with $\lim_{\rho\to\infty} f_{G_{0i}}(\frac{u_i}{\rho\beta_0}) = f_{G_{0i}}(0^+) = c_{0i}$, we obtain

$$\lim_{\rho \to \infty} \rho^2 P_{o,0} = \frac{c_{01}c_{02}}{\beta_0^2}\left(\int_{\frac{1}{2}\log(1+u)\le R} du\right)^2 = \text{Right Hand Side (RHS) of (4)} .$$

We now turn to $P_{o,1}$. Let $P_i(\rho) = \mathbb{P}[\frac{1}{2}\log(1+\rho\beta_0 G_{0i}) \le R]$ be the outage probability of relay R_i. The probability $P_{o,1}$ writes

$$P_{o,1} = \mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{1\}](1-P_1(\rho))P_2(\rho) + \mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{2\}](1-P_2(\rho))P_1(\rho) . \quad (9)$$

The derivation made for $P_{o,0}$ above showed that $P_i(\rho) \sim \frac{1}{\rho}\frac{c_{0i}}{\beta_0}(\exp(2R)-1)$ as $\rho \to \infty$. In particular, this shows that $1-P_i(\rho) \sim 1$ as $\rho \to \infty$. Moreover, when relay R_i is the sole active relay, the associated delay τ_i has no influence on the conditional outage probability $\mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{i\}]$, which writes $\mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{i\}] = \mathbb{P}\left[\frac{1}{2}\log\left(1+\rho\beta_i G_{i3}\right) \le R\right] \sim \frac{1}{\rho}\frac{c_{i3}}{\beta_i}(\exp(2R)-1)$. Getting back to Eq. (9) we deduce from these observations that $\lim_{\rho \to \infty} \rho^2 P_{o,1} = \text{RHS of (5)}$.

We now consider $P_{o,2}$. This probability writes

$$P_{o,2} = \mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{1,2\}](1-P_1(\rho))(1-P_2(\rho)) \sim_{\rho\to\infty} \mathbb{P}[D \text{ fails } / \ \mathcal{D}(s) = \{1,2\}].$$

When the two relays successfully decode the source message, they send this message on the equivalent multipath channel with the frequency selective transfer function $\mathcal{G}(f)$ defined as (see Eq. (1))

$$\mathcal{G}(f) = \left(\sqrt{\beta_1\rho}H_{13}e^{-2\imath\pi f\tau_1} + \sqrt{\beta_2\rho}H_{23}e^{-2\imath\pi f\tau_2}\right)\mathbf{1}_{[-1/2,1/2]}(f).$$

The mutual information $\mathcal{I}(\sqrt{\beta_1\rho}H_{13},\sqrt{\beta_2\rho}H_{23},\Delta)$ associated with this channel is given by

$$\mathcal{I}(\sqrt{\beta_1\rho}H_{13},\sqrt{\beta_2\rho}H_{23},\Delta)=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\log\left(1+|\mathcal{G}\left(f\right)|^2\right)df \tag{10}$$

$$=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\log\left(1+\left|\sqrt{\beta_1\rho}H_{13}+\sqrt{\beta_2\rho}H_{23}e^{-2\imath\pi f\Delta}\right|^2\right)df. \tag{11}$$

Now we have

$$\begin{aligned}\rho^2\mathbb{P}[D\text{ fails }/\ \mathcal{D}(s)=\{1,2\}]&=\rho^2\int_{\{(z_1,z_2)\in\mathbb{C}^2:\,\mathcal{I}(\sqrt{\beta_1\rho}z_1,\sqrt{\beta_2\rho}z_2,\Delta)\le R\}}f_{H_{13}}\left(\mathrm{Re}(z_1),\mathrm{Im}(z_1)\right)f_{H_{23}}(\mathrm{Re}(z_2),\mathrm{Im}(z_2))\,dz_1\,dz_2\\&=\frac{1}{\beta_1\beta_2}\int_{\{(u_1,u_2)\in\mathbb{C}^2:\,\mathcal{I}(u_1,u_2,\Delta)\le R\}}f_{H_{13}}(\mathrm{Re}(u_1)/\sqrt{\beta_1\rho},\mathrm{Im}(u_1)/\sqrt{\beta_1\rho})\times\\&\qquad f_{H_{23}}(\mathrm{Re}(u_2)/\sqrt{\beta_2\rho},\mathrm{Im}(u_2)/\sqrt{\beta_2\rho})\,du_1\,du_2\\&\xrightarrow[\rho\to\infty]{}\frac{b_{13}b_{23}}{\beta_1\beta_2}\int_{\{(u_1,u_2)\in\mathbb{C}^2:\,\mathcal{I}(u_1,u_2,\Delta)\le R\}}du_1\,du_2\\&=\text{RHS of (6) by Lemma }1.\end{aligned}$$

We now prove that ξ_2 is given by Eq. (7) when Δ is a non-zero positive integer $\Delta\in\mathbb{N}^*$ (assuming without loss of generality that the symbol period T_s is equal to one). Writing $H_{i3}=\sqrt{G_{i3}}e^{2\imath\pi\theta_{i3}}$, the mutual information $\mathcal{I}$ given by Eq. (11) becomes:

$$\mathcal{I}=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\log\left(1+\left|\sqrt{\beta_1\rho G_{13}}+\sqrt{\beta_2\rho G_{23}}e^{-2\imath\pi((\theta_{13}-\theta_{23})+f\Delta)}\right|^2\right)df. \tag{12}$$

As the integrand is periodic and spans an integer number of periods on the integration interval $[-1/2,1/2]$, we can assume that $\theta_{13}-\theta_{23}=0$. Moreover, by making the change of variables $t=f\Delta$, we obtain

$$\begin{aligned}\mathcal{I}&=\frac{1}{2\Delta}\int_{-\Delta/2}^{\Delta/2}\log\left(1+\beta_1\rho G_{13}+\beta_2\rho G_{23}+2\rho\sqrt{\beta_1\beta_2G_{13}G_{23}}\cos\left(2\pi t\right)\right)dt\\&=\frac{1}{2}\int_{-1/2}^{1/2}\log\left(1+\beta_1\rho G_{13}+\beta_2\rho G_{23}+2\rho\sqrt{\beta_1\beta_2G_{13}G_{23}}\cos\left(2\pi t\right)\right)dt.\end{aligned}$$

By referring to [15, page 526], this integral admits a closed form solution $\mathcal{I}=\mathcal{I}^{\text{int}}(\beta_1\rho G_{13},\beta_2\rho G_{23})$ where function $\mathcal{I}^{\text{int}}$ is defined over $\mathbb{R}_+^2$ as

$$\mathcal{I}^{\text{int}}(x_1,x_2)=\frac{1}{2}\log\left(\frac{1+x_1+x_2+\sqrt{\left(1+x_1+x_2\right)^2-4x_1x_2}}{2}\right). \tag{13}$$

The outage gain for non-zero positive integer Δ is then derived as

$$\begin{aligned}
\xi_2 &= \lim_{\rho \to \infty} \rho^2 \int_{\{(u_1,u_2)\in\mathbb{R}_+^2:\, \mathcal{I}^{\text{int}}(\beta_1\rho u_1, \beta_2\rho u_2)\leq R\}} f_{G_{13}}(u_1)\, f_{G_{23}}(u_2)\, du_1\, du_2 \\
&= \lim_{\rho \to \infty} \frac{1}{\beta_1\beta_2} \int_{\{(x_1,x_2)\in\mathbb{R}_+^2:\, \mathcal{I}^{\text{int}}(x_1,x_2)\leq R\}} f_{G_{13}}\left(\frac{x_1}{\beta_1\rho}\right) f_{G_{23}}\left(\frac{x_2}{\beta_2\rho}\right) dx_1\, dx_2 \\
&= \frac{c_{13}c_{23}}{\beta_1\beta_2} \int_{\{(x_1,x_2)\in\mathbb{R}_+^2:\, \mathcal{I}^{\text{int}}(x_1,x_2)\leq R\}} dx_1\, dx_2 \ ,
\end{aligned} \tag{14}$$

because $\lim_{\rho\to\infty} f_{G_{13}}(\frac{x_1}{\beta_1\rho}) = f_{G_{13}}(0^+) = c_{13}$ and $\lim_{\rho\to\infty} f_{G_{23}}(\frac{x_2}{\beta_2\rho}) = f_{G_{23}}(0^+) = c_{23}$.

After some simple derivations, we can show that the integration (compact) set $\{(x_1, x_2) \in \mathbb{R}_+^2 : \mathcal{I}^{\text{int}}(x_1, x_2) \leq R\}$ is described by the following equations:

$$\begin{cases} 0 \leq x_1 \leq e^{2R} - 1 \\ 0 \leq x_2 \leq e^{2R} \dfrac{x_1 + 1 - e^{2R}}{x_1 - e^{2R}} \end{cases} .$$

Thus we have after a simple derivation

$$\int_{\{(x_1,x_2)\in\mathbb{R}_+^2:\mathcal{I}^{\text{int}}(x_1,x_2)\leq R\}} dx_1\, dx_2 = e^{2R}\left(e^{2R} - 2R - 1\right) \ , \tag{15}$$

which leads to Eq. (7).

It remains to prove the Inequalities (8) useful in the general case $\Delta \in \mathbb{R}_+^*$. Getting back to the expression (11) of the mutual information $\mathcal{I}$ associated with channel $\mathcal{G}(f)$, and making the change of variable $t = f\Delta$, we obtain

$$\mathcal{I} = \frac{1}{2\Delta} \int_{-\Delta/2}^{\Delta/2} \log\left(1 + \left|\sqrt{\beta_1\rho} H_{13} + \sqrt{\beta_2\rho} H_{23} e^{-2\imath\pi t}\right|^2\right) dt = \frac{1}{2\Delta}\int_{-\Delta/2}^{\Delta/2} \varphi(t)\, dt.$$

As the integrand $\varphi(t)$ is nonnegative, we have

$$\frac{1}{2\Delta}\int_{-\lfloor\Delta\rfloor/2}^{\lfloor\Delta\rfloor/2} \varphi(t)\, dt \leq \mathcal{I} \leq \frac{1}{2\Delta}\int_{-\lceil\Delta\rceil/2}^{\lceil\Delta\rceil/2} \varphi(t)\, dt \ . \tag{16}$$

By making the change of variable $u = t/\lfloor\Delta\rfloor$ and by using the same argument as the one that follows Eq. (12), we obtain

$$\frac{1}{2\Delta}\int_{-\lfloor\Delta\rfloor/2}^{\lfloor\Delta\rfloor/2} \varphi(t)\, dt = \frac{\lfloor\Delta\rfloor}{2\Delta}\int_{-1/2}^{1/2} \varphi(\lfloor\Delta\rfloor u)\, du = \frac{\lfloor\Delta\rfloor}{\Delta}\mathcal{I}^{\text{int}}\left(\beta_1\rho G_{13}, \beta_2\rho G_{23}\right).$$

Similarly, the upper bound in (16) is equal to $\frac{\lceil\Delta\rceil}{\Delta}\mathcal{I}^{\text{int}}\left(\beta_1\rho G_{13}, \beta_2\rho G_{23}\right)$. By applying to these two bounds the same argument as the one that follows (14) (the integration sets being $\{\mathcal{I}^{\text{int}} \leq (\Delta/\lceil\Delta\rceil)R\}$ and $\{\mathcal{I}^{\text{int}} \leq (\Delta/\lfloor\Delta\rfloor)R\}$) we obtain imediately the bounds specified by (8).

Proposition 1 is proven. ■

Comments on Proposition 1: When the two relays use the same codebook, Proposition 1 shows that introducing a delay between the two relays is beneficial in terms of diversity. This proposition shows in particular that the outage gain does not depend on the relative delay Δ when Δ is a non-zero integer. In the general situation where $\Delta \in \mathbb{R}_+^*$, the bounds provided by Inequalities (8) become quite close to each other and to the integer delay outage gain when Δ is of the order of a few symbol periods. We note that another lower bound for real relative delays was derived in [16].

Another point is that, using the same codebook, we have a diversity of two only when the delay between the relays is non zero and the diversity becomes equal to one if $\Delta = 0$. To see this, assume that $\tau_1 = \tau_2 = \tau$ where τ is a given common delay in Eq. (1). In the case where both relays succeed in decoding ($\mathcal{D}_s = \{R_1, R_2\}$), the signal received by the destination during the second time slot becomes

$$y_{DF}\,(t) = \sqrt{\rho}\left(\sqrt{\beta_1}\,H_{13}\;+\sqrt{\beta_2}\,H_{23}\right)x(k)\Phi(t-\tau-k)\;+\;n_3(t)\;.$$

The equivalent channel $\left(\sqrt{\beta_1}\,H_{13}\;+\sqrt{\beta_2}\,H_{23}\right)$ generates a diversity of one only. This can be easily seen in *e.g.* the Rayleigh case. The lost of diversity for $\Delta = 0$ can also be seen by simulations in Figure 2.

It is interesting to compare the outage gain of this asynchronous DF protocol with the more classical synchronous DF protocol where the two relays use independent codebooks. In Section VII, it is shown by simulation that the two protocols have nearly the same performance.

To better explain the difference between the two DF protocols above, let us consider a simple example. Suppose that the source S transmits the following N-symbol frame $[S_1, S_2, ..., S_N]$ using the codebook C and that the relays R_1 and R_2 successfully decode the entire frame. In the asynchronous DF protocol, R_1 and R_2 use the same codebook C' to decode the frame in the second phase; so the two relays send $[S_1', S_2', ..., S_N']$ to the destinaton D with a delay Δ between the two transmissions. On the contrary, for the classical synchronous DF protocol, relays R_1 and R_2 use respectively the codebooks C^1 and C^2 for example to decode in the second phase and they send respectively the frames $[S_1^1, S_2^1, ..., S_N^1]$ and $[S_1^2, S_2^2, ..., S_N^2]$ synchronously to D.

IV. Outage gain of the AF Protocol

We now consider the AF protocol. We begin with some observations about the choice of the power gains A_i (see Eq.(2)) used by the relays. Two strategies exist: either a fixed gain strategy is chosen where A_i is set to a predetermined value $A_i = \beta_i$, or a variable gain strategy is adopted where A_i depends on G_{0i} according to $A_i = \dfrac{\beta_i \rho}{\beta_0 \rho\, G_{0i} + 1}$. The latter strategy guarantees that the power spent by the relay is equal to $\beta_i \rho$ whatever is the value of G_{0i} [17]. The two strategies will be considered herein.

Let us begin by providing the general expression of the outage gain ξ_{AF}, which is given as in the DF case by $\xi_{AF} = \lim_{\rho\to\infty} \rho^2 P_o$. Due to the different time delays of the relay signals, an equivalent multipath fading channel appears between the relays and the destination. The associated mutual information is $\mathcal{I}_{AF} = \int_{-1/2}^{1/2} \log\left(1 + \dfrac{S_x(f)}{S_n}\right) df$ where $S_x(f)$ is the Power Spectral Density (PSD) of the information signal received by the destination and S_n is the noise PSD at the destination; Using Eq. (2), these two quantities are given by:

$$S_x(f) = \left| \sqrt{A_1 \beta_0 \rho} H_{01} H_{13} e^{-2\imath\pi f \tau_1} + \sqrt{A_2 \beta_0 \rho} H_{02} H_{23} e^{-2\imath\pi f \tau_2} \right|^2,$$

$$S_n = A_1 G_{13} + A_2 G_{23} + 1.$$

In order to study ξ_{AF}, we restrict ourselves in most of this section to the case where $\Delta = \tau_2 - \tau_1 \in \mathbb{N}^*$. In the case $\Delta \in \mathbb{R}_+^*$ results similar to the DF case will be given succinctly at the end of this section. By a derivation similar to the DF case above for Δ integer (argument developed between Eqs (12) and (13)), the mutual information $\mathcal{I}_{AF}$ writes

$$\begin{aligned}
\mathcal{I}_{AF} &= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \left|\sqrt{x_1} + \sqrt{x_2}\, e^{2\imath\pi f \Delta}\right|^2\right) df \\
&= \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + x_1 + x_2 + 2\sqrt{x_1 x_2} \cos(2\pi f)\right) df = \mathcal{I}^{\text{int}}(x_1, x_2)
\end{aligned}$$

with

$$x_1 = \frac{A_1 \beta_0 \rho\, G_{01} G_{13}}{A_1\, G_{13} + A_2\, G_{23} + 1}, \quad x_2 = \frac{A_2 \beta_0 \rho\, G_{02} G_{23}}{A_1 G_{13} + A_2 G_{23} + 1} \tag{17}$$

and $\mathcal{I}^{\text{int}}(x_1, x_2)$ is given by Eq. (13). We denote by $\mathcal{F}_{\text{fix}}(G_{01}, G_{02}, G_{13}, G_{23}, \rho)$ the function $\mathcal{F}_{\text{fix}}(G_{01}, G_{02}, G_{13}, G_{23}, \rho) = \mathcal{I}^{\text{int}}(x_1, x_2)$ where (x_1, x_2) are replaced with their values in (17) and $A_i = \beta_i$. Similarly, we let $\mathcal{F}_{\text{var}}(G_{01}, G_{02}, G_{13}, G_{23}, \rho) = \mathcal{I}^{\text{int}}(x_1, x_2)$ where this time the A_i are the variable gains $A_i = \beta_i \rho / (\beta_0 \rho G_{0i} + 1)$. With these notations, ξ_{AF} can be written as

$$\xi_{AF} = \lim_{\rho\to\infty} \rho^2 \int_{\mathcal{F}_{\text{fix}} \text{ or } \mathcal{F}_{\text{var}}(u_1,u_2,u_3,u_4,\rho)\le R} f_{G_{01}}(u_1)\, f_{G_{02}}(u_2)\, f_{G_{13}}(u_3)\, f_{G_{23}}(u_4)\ \ du_1 du_2 du_3 du_4. \quad (18)$$

To derive this expression, we have to delineate the domain of variation of the four parameters (u_1, u_2, u_3, u_4) for large ρ then solve a complicated integral. In general, this task shows to be very difficult. In order to simplify the problem, we shall consider three interesting particular cases for which this limit is tractable. We shall consider in turn 1) The case where the source-relay channels are Gaussian, 2) The case where the relay-destination channels are Gaussian, and finally 3) The performance bound obtained when the relays are noiseless.

A. Outage gain for some particular cases

1) Gaussian source-relay links: We consider that the channels from the source to the relays $S \to R_1$ and $S \to R_2$ are Gaussian channels, in other words, H_{01} and H_{02} are deterministic. This case corresponds to a downlink mobile communication with fixed position relays and the existence of a line of sight between the Base Station (the source) and the two relays. In the sequel, we shall denote the power gains of the deterministic channels by lower case letters. Hence the power gains of the $S \to R_1$ and and $S \to R_2$ channels will be denoted g_{01} and g_{02} respectively.

Proposition 2: Consider the asynchronous network and the AF protocol described in Section II. For Gaussian source-relay links, given a rate R and assuming a fixed gain strategy, the outage probability P_o satisfies

$$\rho^2 P_o \xrightarrow[\rho\to\infty]{} \xi_{AF} = \frac{c_{13}c_{23}}{\beta_0^2 \beta_1 \beta_2 g_{01} g_{02}}\, e^{2R}\left(e^{2R} - 2R - 1\right). \quad (19)$$

Assuming a variable gain strategy, the following holds true:

$$\rho^2 P_o \xrightarrow[\rho\to\infty]{} \xi_{AF} = \frac{c_{13}c_{23}}{\beta_1 \beta_2}\, e^{2R}\left(e^{2R} - 2R - 1\right). \quad (20)$$

Sketch of proof: In the setting of this proposition, the RHS of (18) writes

$$\lim_{\rho\to\infty} \rho^2 P_o = \lim_{\rho\to\infty} \int_{\mathcal{F}_{\text{fix}} \text{ or } \mathcal{F}_{\text{var}}(g_{01},g_{02},u/\rho,v/\rho,\rho)\le R} f_{G_{13}}(u/\rho)\, f_{G_{23}}(v/\rho)\ \ du\, dv.$$

Let us begin with the fixed gain relays case ($A_i = \beta_i$). The idea of the proof is the following: for large ρ, we have $x_1 \sim \beta_0\beta_1 g_{01}\rho u$ and $x_2 \sim \beta_0\beta_2 g_{02}\rho v$ in (17). Furthermore, $f_{G_{13}}(u/\rho) \sim c_{13}$ and $f_{G_{23}}(v/\rho) \sim c_{23}$, hence one expects that

$$\lim_{\rho\to\infty} \rho^2 P_o = c_{13}c_{23} \int_{\mathcal{I}^{\text{int}}(\beta_0\beta_1 g_{01}u, \beta_0\beta_2 g_{02}v)\le R} du\, dv = \frac{c_{13}c_{23}}{\beta_0^2\beta_1\beta_2 g_{01}g_{02}} \int_{\mathcal{I}^{\text{int}}(u,v)\le R} du\, dv = \text{RHS of (19)}$$

by Eq. (15). Heuristically, these derivations show that an outage event occurs when both gains G_{13} and G_{23} are small (of order $1/\rho$).

A rigorous proof for (19) involves a rather lengthy Dominated Convergence Theorem argument which role is to justify the exchange between $\lim\limits_{\rho \to\infty}$ and $\int$. These details are omitted for lack of space.

In the variable gain relays case, we have $x_1 \sim \beta_1\rho\, u$ and $x_2 \sim \beta_2\rho\, v$ for large ρ. A similar derivation to the fixed gain case gives (20). ■

2) Gaussian relay-destination links: Here, we consider that the channels from the relays to the destination $R_1 \to D$ and $R_2 \to D$ are Gaussian channels; thus H_{13} and H_{23} are deterministic and we denote the gains G_{13} and G_{23} as g_{13} and g_{23} respectively. This case corresponds to an uplink mobile communication with fixed position relays and the existence of a line of sight between the relays and the Base Station (the destination).

For fixed gain relays, we have

$$x_1 = \frac{\beta_1\beta_0\rho\, g_{13}G_{01}}{\beta_1 g_{13} + \beta_2 g_{23} + 1}, \quad x_2 = \frac{\beta_2\beta_0\rho\, g_{23}G_{02}}{\beta_1 g_{13} + \beta_2 g_{23} + 1}.$$

Similarly to the previous section, the RHS of (18) writes

$$\begin{aligned}
\lim_{\rho\to\infty} \rho^2 P_o &= \lim_{\rho\to\infty} \int_{\mathcal{F}_{\text{fix}}(g_{13}, g_{23}, u/\rho, v/\rho, \rho)\le R} f_{G_{01}}(u/\rho)\, f_{G_{02}}(v/\rho)\ \, du\, dv \\
&= \frac{(\beta_1 g_{13} + \beta_2 g_{23} + 1)^2\, c_{01}c_{02}}{\beta_0^2\beta_1\beta_2\, g_{13}g_{23}} \int_{\mathcal{I}^{\text{int}}(u,v)\le R} du\, dv \\
&= \frac{(\beta_1 g_{13} + \beta_2 g_{23} + 1)^2\, c_{01}c_{02}}{\beta_0^2\beta_1\beta_2\, g_{13}g_{23}}\, e^{2R}\left(e^{2R} - 2R - 1\right). \qquad (21)
\end{aligned}$$

In the variable gain relays case, it is difficult to obtain simple closed form expressions for the outage gain. This case will be treated by simulations in Section VII.

3) Noiseless relay bound: Now, we consider that the noise at the relays level is null ($n_1(t) = n_2(t) = 0$). Moreover, we assume that all channels are Rayleigh channels ($H_{ij} \sim \mathcal{CN}(0, \sigma_{ij}^2)$). Clearly, the outage gain in this case is a lower bound of the general case. Let us focus on the fixed gain relays case. In this case, it is clear that $x_1 = \beta_1\beta_0\rho G_{01}G_{13}$ and $x_2 = \beta_2\beta_0\rho G_{02}G_{23}$, which leads us to consider the density function of the product of two independent exponential random variables. Such densities are considered in [17]. As demonstrated in Appendix B, the densities $f_{G_{01}G_{13}}$ and $f_{G_{02}G_{23}}$ satisfy in a neighborhood of $\rho = \infty$:

$$f_{G_{01}G_{13}}(u/\rho) \sim c_{01}c_{13}\ln\rho; \quad f_{G_{02}G_{23}}(v/\rho) \sim c_{02}c_{23}\ln\rho.$$

Due to the $\ln\rho$ factor in these equations, $f(\rho)P_o$ does not converge to a finite value when $f(\rho) = \rho^2$. In order to obtain a meaningful definition of the outage gain in this case, we need to take $f(\rho) = (\frac{\rho}{\ln(\rho)})^2$ instead. With this definition, we have

$$\begin{aligned}\xi^n_{AF_{\text{fix}}} &= \lim_{\rho\to\infty}\frac{\rho^2}{(\ln\rho)^2}\int_{\mathcal{F}_{\text{fix}}(u,v,\rho)\le R} f_{G_{01}G_{13}}(u)\, f_{G_{02}G_{23}}(v)\, du\, dv \\ &= \frac{c_{01}c_{13}c_{02}c_{23}}{\beta_0^2\beta_1\beta_2}\int_{\mathcal{I}^{\text{int}}(u,v)\le R} du\, dv \\ &= \frac{c_{01}c_{13}c_{02}c_{23}}{\beta_0^2\beta_1\beta_2}\, e^{2R}\left(e^{2R} - 2R - 1\right). \end{aligned} \tag{22}$$

In the variable gain relays case, we have $x_1 = \frac{\beta_0\beta_1\rho^2 G_{01}G_{13}}{\beta_0\rho G_{01}+1}$ and $x_2 = \frac{\beta_0\beta_2\rho^2 G_{02}G_{23}}{\beta_0\rho G_{02}+1}$. The densities of these two random variables are continuous at 0^+ with a limit independent of ρ: $f_{G_{01}G_{13}}(u/\rho) \sim c_{13}$ and $f_{G_{02}G_{23}}(v/\rho) \sim c_{23}$ (see Appendix B). In this situation, the outage gain $\xi^n_{AF_{\text{var}}}$ is associated with the usual $f(\rho) = \rho^2$, *i.e.*, $\xi^n_{AF_{\text{var}}} = \lim_{\rho\to\infty}\rho^2 P_o$. After some similar derivations like for the fixed gain relays, we obtain

$$\xi^n_{AF_{\text{var}}} = \frac{c_{13}c_{23}}{\beta_0^2\beta_1\beta_2}\, e^{2R}\left(e^{2R} - 2R - 1\right). \tag{23}$$

B. Real relative delays

In Section IV-A, we can notice that the outage gains (19), (20), (21), (22) and (23) calculated for $\Delta \in \mathbb{N}^*$ are independent from the value of Δ. In this section, we give lower and upper bounds for the outage gains of the AF protocol particular cases for non-zero positive real relative delays.

Similarly to the DF protocol in Section III, to obtain lower and upper bounds of the outage gains for the AF particular cases considered above, we use the same derivations as in the proof of Equation (8) in Proposition 1. Therefore, to obtain lower bounds for $\Delta \in \mathbb{R}_+^*$ we replace, in the expressions of the outage gains obtained for integer relative delays in Section IV-A (Equations (19), (20), (21), (22) and (23)), R with $R\frac{\Delta}{\lceil\Delta\rceil}$ and to obtain upper bounds for real $\Delta \geq 1$ we replace R with $R\frac{\Delta}{\lfloor\Delta\rfloor}$ in the same expressions of outage gains.

V. Power Distribution Optimization

This section is devoted for the optimization of the power ditribution between the source and the two relays based on the mimimization of the outage gain. Let us begin by evaluating the total average power expenditure in the DF case, the AF case with variable gains and the AF case with fixed gain. Choosing $\beta_0 + \beta_1 + \beta_2 = 1$ and recalling that any node is active at most half of the time, the total average power in the DF case is

$$\frac{\beta_0 \rho}{2} + \frac{\beta_1 \rho}{2}\mathbb{P}\left[R_1 \in \mathcal{D}(s)\right] + \frac{\beta_2 \rho}{2}\mathbb{P}\left[R_2 \in \mathcal{D}(s)\right] \lesssim \frac{\rho}{2} \quad \text{at high SNR}$$

because $\mathbb{P}\left[R_i \in \mathcal{D}(s)\right] \approx 1$ at high SNR. In the AF case with variable relay gain, the total transmitted power is simply $\frac{1}{2}\left(\beta_0 + \beta_1 + \beta_2\right)\rho = \frac{\rho}{2}$ with $\beta_0 + \beta_1 + \beta_2 = 1$. In the AF case with fixed gain, the transmitted power is

$$\frac{1}{2}\left[\beta_0 \rho + \beta_1\left(\rho \mathbb{E}G_{01} + 1\right) + \beta_2\left(\rho \mathbb{E}G_{02} + 1\right)\right] < \frac{\rho}{2}\left[\beta_0 + \beta_1 \mathbb{E}G_{01} + \beta_2 \mathbb{E}G_{02}\right] \ .$$

Our purpose is to minimize ξ with respect to $(\beta_0, \beta_1, \beta_2)$. In all the considered cases, the outage gain $\xi = \xi(\beta_0, \beta_1, \beta_2)$ is a convex function. This is due to the fact that it is a sum of functions of the type $f(\beta_0, \beta_1, \beta_2) = K\beta_0^{-a}\beta_1^{-b}\beta_2^{-c}$ where a, b, c and K are non negative constants (see for instance Eqs (4)-(6)). By deriving the Hessian matrix, such functions can be shown to be convex. The constraint set in *e.g.* the DF case and the AF with variable gain case is $\{(\beta_0, \beta_1, \beta_2) \in \mathbb{R}_+^3 : \beta_0 + \beta_1 + \beta_2 = 1\}$ and therefore is also convex. The minimization can be done easily for instance with the help of a descent method.

Some examples of channel distributions and delay profiles are considered in Section VII; in these examples, we include a discussion on the power optimization and on the SNR gain that comes out of this optimization in comparison with the equal power distribution. Also, some insights on the optimal power allocation are given.

VI. Diversity Multiplexing Tradeoff

The Diversity-Multiplexing Tradeoff (DMT) reveals a fundamental relationship between the diversity gain which characterizes the asymptotic rate of decoding error approaching zero as SNR increases, and the multiplexing gain which characterizes the asymptotic spectral efficiency in the large SNR regime [18]. The DMT function $d(r)$ is associated with the outage probability P_o as $d\left(r\right) = -\lim\limits_{\rho \rightarrow \infty} \dfrac{\log\ P_o(R)}{\log \rho}$ where R is the data rate, assumed to increase with the SNR ρ as $R = r \log \rho$. Here $d(r)$ is called the diversity gain and the factor r is called the multiplexing gain.

Proposition 3: The DMT of the considered asynchronous two-relay two-hop wireless network for the DF and AF protocols for $r \in \left[0, \frac{1}{2}\right)$ *is*

$$d\left(r\right) = 2\left(1 - 2\,r\right)\mathbf{1}_{[0,1/2]}(r)\ . \tag{24}$$

Proof: Let us consider the DF protocol. Recall that the general form of the outage probability for this protocol is given by Eq. (3). Here the DMT function is given by $d(r) = \min(d_0(r), d_1(r), d_2(r))$ where $d_i(r) = -\lim_\rho \log P_{o,i}(R) / \log \rho$ for $i = 0, 1, 2$. By a standard derivation [3], we have $d_0(r) = d_1(r) = 2\left(1 - 2r\right)\mathbf{1}_{[0,1/2]}(r)$.
Let us consider $d_2(r) = \lim_\rho \log \mathbb{P}\left[\mathcal{I} \leq r \log \rho\ /\ \ |\mathcal{D}(s)| = 2\right] / \log \rho$ where $\mathcal{I}$ is given by Eq. (11). As $\log$ is a concave function, we have (see Eq. (12))

$$\begin{aligned}\mathcal{I} &\leq \frac{1}{2}\log\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(1 + \left|\sqrt{\beta_1\rho G_{13}} + \sqrt{\beta_2\rho G_{23}}e^{-2\imath\pi((\theta_{13}-\theta_{23})+f\Delta)}\right|^2\right)df \\ &\leq \frac{1}{2}\ \log\left(1 + 2\left(\rho\beta_1 G_{13} + \rho\beta_2 G_{23}\right)\right).\end{aligned}$$

By deriving the DMT on the RHS of this expression [3], we obtain $d_2(r) \leq 2\left(1 - 2r\right)\mathbf{1}_{[-1/2,1/2]}(r)$.
We now look for a lower bound on $d_2(r)$. The derivations are inspired by those made by Grokop and Tse for the Inter Symbol Interference channel [19], [20]. The starting point is the expression

(10) of $\mathcal{I}$. Let $\|\mathcal{G}\|^2 = \int_{-1/2}^{1/2} |\mathcal{G}(f)|^2 df$, and write $\widetilde{\mathcal{G}}(f) = \frac{\mathcal{G}(f)}{\|\mathcal{G}\|}$. Fixing $\epsilon > 0$, we have:

$$\log\left(1 + |\mathcal{G}(f)|^2\right) = \log\left(1 + \|\mathcal{G}\|^2 \left|\widetilde{\mathcal{G}}(f)\right|^2\right)$$
$$= \log\left(1 + \|\mathcal{G}\|^2 \left|\widetilde{\mathcal{G}}(f)\right|^2\right) \times \mathbf{1}_{\left|\widetilde{\mathcal{G}}(f)\right| \geq \epsilon} + \log\left(1 + \|\mathcal{G}\|^2 \left|\widetilde{\mathcal{G}}(f)\right|^2\right) \times \mathbf{1}_{\left|\widetilde{\mathcal{G}}(f)\right| < \epsilon}.$$

Hence

$$\begin{aligned}
\mathcal{I} &= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + |\mathcal{G}(f)|^2\right) df \geq \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \|\mathcal{G}\|^2 \left|\widetilde{\mathcal{G}}(f)\right|^2\right) \times \mathbf{1}_{\left|\widetilde{\mathcal{G}}(f)\right| \geq \epsilon}\, df \\
&\geq \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(1 + \|\mathcal{G}\|^2 \epsilon^2\right) \times \mathbf{1}_{\left|\widetilde{\mathcal{G}}(f)\right| \geq \epsilon}\, df = \frac{1}{2}\left(1 - \left|\mathcal{U}\left(\widetilde{\mathcal{G}}, \epsilon\right)\right|\right) \log\left(1 + \|\mathcal{G}\|^2 \epsilon^2\right)
\end{aligned}$$

with $\mathcal{U}\left(\widetilde{\mathcal{G}}, \epsilon\right) = \left\{f \in \left[-\frac{1}{2}, \frac{1}{2}\right) : \left|\widetilde{\mathcal{G}}(f)\right| < \epsilon\right\}$ and $|\mathcal{U}|$ is the Lebesgue measure of $\mathcal{U}$.

By consequence, the outage probability $\mathbb{P}\left[\mathcal{I} \leq r\log\rho \;/\; |\mathcal{D}(s)| = 2\right]$ satisfies

$$\mathbb{P}\left[\mathcal{I} \leq r\log\rho \;/\; |\mathcal{D}(s)| = 2\right] \leq \mathbb{P}\left[\log\left(1 + \|\mathcal{G}\|^2 \epsilon^2\right) \leq \frac{2r\log\rho}{1 - \left|\mathcal{U}\left(\widetilde{\mathcal{G}}, \epsilon\right)\right|}\right]. \tag{25}$$

By a technique similar to [20, pages 56-57], it is possible to show that

$$\forall\epsilon > 0, \exists\mu > 0 \text{ such that } \sup_{\widetilde{\mathcal{G}}(f):\|\widetilde{\mathcal{G}}\| \leq 1} \left|\mathcal{U}\left(\widetilde{\mathcal{G}}, \epsilon\right)\right| < \mu,$$

which results in

$$\sup_{\widetilde{\mathcal{G}}(f):\|\widetilde{\mathcal{G}}\| \leq 1} \frac{2r\log\rho}{1 - \left|\mathcal{U}\left(\widetilde{\mathcal{G}}, \epsilon\right)\right|} < \frac{2r\log\rho}{1 - \mu}. \tag{26}$$

Moreover, we have

$$\begin{aligned}
\|\mathcal{G}\|^2 &= \rho\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\sqrt{\beta_1}H_{13} + \sqrt{\beta_2}H_{23}e^{-2\imath\pi f\Delta}\right|^2 df \\
&= \rho\left(\beta_1|H_{13}|^2 + \beta_2|H_{23}|^2 + 2\sqrt{\beta_1\beta_2}\,\frac{\sin\pi\Delta}{\pi\Delta}\,\mathrm{Re}(H_{13}H_{23}^*)\right) \\
&= \rho\left(\left(1 - \frac{\sin\pi\Delta}{\pi\Delta}\right)\left(\beta_1|H_{13}|^2 + \beta_2|H_{23}|^2\right) + \frac{\sin\pi\Delta}{\pi\Delta}\left|\sqrt{\beta_1}H_{13} + \sqrt{\beta_2}H_{23}\right|^2\right) \\
&\geq \rho\left(1 - \left|\frac{\sin\pi\Delta}{\pi\Delta}\right|\right)\left(\beta_1|H_{13}|^2 + \beta_2|H_{23}|^2\right) = \rho K(\Delta)\left(\beta_1 G_{13} + \beta_2 G_{23}\right) \qquad (27)
\end{aligned}$$

where $K(\Delta) > 0$ for $\Delta > 0$. Plugging Inequalities (26) and (27) into (25), we obtain

$$\mathbb{P}\left[\mathcal{I} \leq r\log\rho \;/\; |\mathcal{D}(s)| = 2\right] \leq \mathbb{P}\left[\log\left(1 + \rho\epsilon^2 K(\Delta)\left(\beta_1 G_{13} + \beta_2 G_{23}\right)\right) \leq \frac{2r\log\rho}{1 - \mu}\right].$$

Hence the bound is $d_2(r) \geq 2\left(1 - \frac{2r}{1-\mu}\right)$. Letting $\mu \to 0$, we obtain $d_2(r) \geq 2(1-2r)$. Combining with the upper bound, we end up with $d_2(r) = 2(1-2r)$, which proves (24) for the DF protocol.

The DMT derivation for the AF protocol can be done similarly. ■

VII. Numerical Results and Interpretations

Our first simulation results (Fig. 2) concern the performance of the DF protocol. In this figure, all channels are Rayleigh fading with unit variance ($\sigma_{ij}^2 = 1$), and the data rate has been set to $R = 1$. Moreover, all nodes have the same power. In this figure, the outage approximation $P_o \approx \xi_{DF}\rho^{-2}$ given by Prop. 1 is plotted for integer values of Δ. Simulation results for different values of Δ are also shown. One can notice a very good fit between the approximation $P_o \approx \xi_{DF}\rho^{-2}$ and the simulation results for high SNR regime. For comparison purposes, we test on the same figure the outage probability of a synchronous DF protocol which uses independent codebooks at the relays (see comments after Prop. 1). We notice that the performance of the asynchronous DF protocol is quite comparable to the performance of the synchronous one. We can also remark that the outage gain becomes invariant and equal to the case of an integer relative delay when Δ is sufficiently high. Therefore, to insure good outage probability performance, the relays can introduce additional random delays before transmitting in the second phase.

The AF protocol is tested in Figures 3 and 4. The simulation conditions are identical to those of Figure 2 as concerns the rate, the powers and the Rayleigh channels. In Figure 3, we plot the outage probability for the first two particular cases considered in Section IV-A: Gaussian source-relay channels and Gaussian relay-destination channels. Fixed gains as well as variable gains have been considered at the relays. For the case where the source-relay channels are Gaussian, we take $g_{01} = 0.6$ and $g_{02} = 1$ while for the Gaussian relay-destination channels case, we assume that $g_{13} = 0.6$ and $g_{23} = 1$. We can notice that the outage performance in the Gaussian S-R case is better than the Gaussian R-D case for the two types of relays which is obvious because in the first case the signals received by the relays are not attenuated.

In Figure 4, the outage probabilities for the AF protocol in the “all Rayleigh” case is simulated and compared to the noiseless relay bound (Section IV-A3). In these schemes, all the channels in phase I and phase II are Rayleigh fading with unit variances. Although for the noiseless case the outage performance is better for the variable gain relays as concluded in Section IV-A3, in the general case with the presence of noise at the relays level, the outage probability of fixed gain relays outperforms the variable gain relays one. This result is due to the high noise amplification in the case of variable gain relays when the source-relay channels are weak.

In Figure 5, we compare the outage probability performances of the DF and AF protocols for fixed gain relays in the general case (all the channels are Rayleigh fading with unit variance) and for Gaussian source-relay links $(g_{01} = g_{02} = 1)$ with equal power distribution. We can notice that the DF protocol outperforms the AF protocol in the general case. But, in the particular case of Gaussian source-relay channels and for high SNR values, the two protocols have the same outage performance. So in this case we can use the AF protocol to reduce the hardware complexity demanded at the relays for the DF protocol.

In Figure 6, we illustrate the SNR gain due to our power optimization approach for the DF protocol. Rayleigh channels with variances $(\sigma_{01}^2, \sigma_{02}^2, \sigma_{13}^2, \sigma_{23}^2) = (0.5, 1, 5, 2)$ are adopted. The lower and upper bounds and the simulations of the outage probability for $R = 1$ and $\Delta = 1.8T_s$ are plotted for both equal and optimized power distribution. In this situation, the power optimization results in a SNR gain of about 1.2 dB. Figure 8 shows the SNR gain due optimization of the outage gain in the DF case with respect to the relative distance between the relays and the source to the distance between the source and the destination. We adopt the model of network shown in Figure 7 where the relays R_1 and R_2 are located on the source-destination link axis with the same distance to the source. A delay of one symbol period is introduced between the two relay signals. The channels are Rayleigh channels with the power decay profile $\sigma_{ij} \propto d_{ij}^{-3}$ with d_{ij} is the distance between nodes i and j. The plain curve represents the theoretical SNR gain obtained by optimizing the β_is (based on outage gains) while the dashed curve represents the SNR gain obtained by simulation for an outage probability set to 10^{-3}. We notice that the optimization is all the more useful as the relays lie on the extremes of the S-D axis.

VIII. Conclusions

In this paper, the outage gain of an asynchronous two-relay network using the Decode-and-Forward and Amplify-and-Forward protocols is calculated for different values of relative delay Δ between the two relays and for different source transmission rate R. Besides, the DMT of the considered network model is proved to be equal to $d(r) = 2(1-2r)$ for DF and AF cases. Simulation results of the outage probability for different cases confirmed the theoretical calculations and the merit of the power distribution optimization method is also illustrated.

Appendix A
Proof of Lemma 1

Consider the family of functions $\phi_\epsilon(x) = \mathbf{1}_{[0,\epsilon]}(x)$. Denote by $B(0,\sqrt{\epsilon})$ the ball of $\mathbb{R}^2$ centered at zero with radius $\sqrt{\epsilon}$. We have $\lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[\phi_\epsilon(|H_{ij}|^2)] = \lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\int \phi_\epsilon(x^2+y^2)f_{H_{ij}}(x,y)dx\,dy = \lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\int_{B(0,\sqrt{\epsilon})} f_{H_{ij}}(x,y)dx\,dy$. As $f_{H_{ij}}$ is continuous at the point $(0,0)$, we have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{B(0,\sqrt{\epsilon})} f_{H_{ij}}(x,y)dx\,dy = f_{H_{ij}}(0,0)\lim_{\epsilon\to 0}\frac{\text{Volume}\,(B(0,\sqrt{\epsilon}))}{\epsilon} = \pi b_{ij}\ .$$

We can also write $\mathbb{E}[\phi_\epsilon(|H_{ij}|^2)] = \mathbb{E}[\phi_\epsilon(G_{ij})] = \int \phi_\epsilon(u) f_{G_{ij}}(u)du = \int_0^\epsilon f_{G_{ij}}(u)du$. As $f_{G_{ij}}$ is right continuous at zero, we have $\lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}[\phi_\epsilon(|H_{ij}|^2)] = \lim\limits_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^\epsilon f_{G_{ij}}(u)du = f_{G_{ij}}(0^+)$. Therefore, $c_{ij} = \pi b_{ij}$.

Appendix B
Behavior of the Product Channel Density Function

Consider the fixed gain relay fading channel $h = h_1 h_2$, with respective powers σ_1^2 and σ_2^2. The probability density function (pdf) of the the squared envelope $G_{\text{fix}} = |h|^2 = G_1 G_2$ can be deducted from [17] by a simple variable change:

$$f_{G_{\text{fix}}}(g) = 2c_1c_2\; K_0\left(2\sqrt{c_1c_2\, g}\right), \tag{28}$$

where $c_1 = \sigma_1^{-2}$, $c_2 = \sigma_2^{-2}$ and $K_0(.)$ is the zeroth order modified Bessel function of the second kind. Using the developpement of the function K_0 in [15, page 909], we obtain the value of (28) when $\rho \to \infty$:

$$f_{G_{\text{fix}}}\left(g \propto \frac{1}{\rho}\right) = f_{G_{\text{fix}}}\left(0^+\right) = c_1 c_2 \, \ln \rho.$$

For a relay with variable gain, the pdf of the squared envelope of the overall relay channel $G_{\text{var}} = \frac{\rho G_1 G_2}{\rho G_1 + 1}$ can also be deducted from [17] by a simple variable change:

$$f_{G_{\text{var}}}(g) = 2c_2 \, e^{-c_2 g} \left[\sqrt{\frac{c_1 c_2 g}{\rho}} K_1\left(2\sqrt{\frac{c_1 c_2 g}{\rho}}\right) + c_1 K_0\left(2\sqrt{\frac{c_1 c_2 g}{\rho}}\right)\right], \tag{29}$$

where $K_1(.)$ is the first-order modified Bessel function of the second kind. Using the developpement of the functions K_0 and K_1 in [15, page 909], we obtain the value of (29) when $\rho \to \infty$:

$$f_{G_{\text{var}}}\left(g \propto \frac{1}{\rho}\right) = f_{G_{\text{var}}}\left(0^+\right) = c_2.$$

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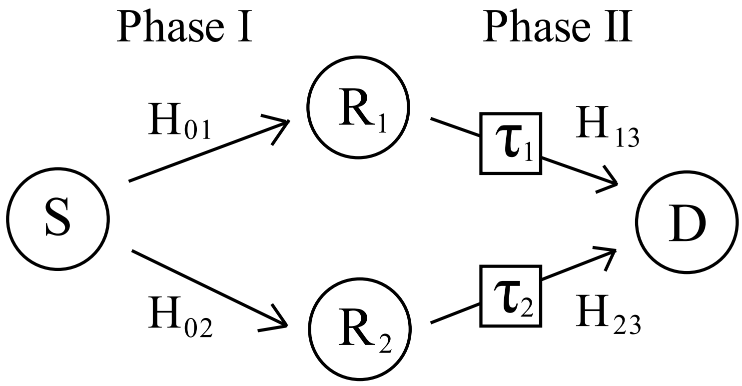

Fig. 1. The asynchronous two-relay two-hop wireless network model.

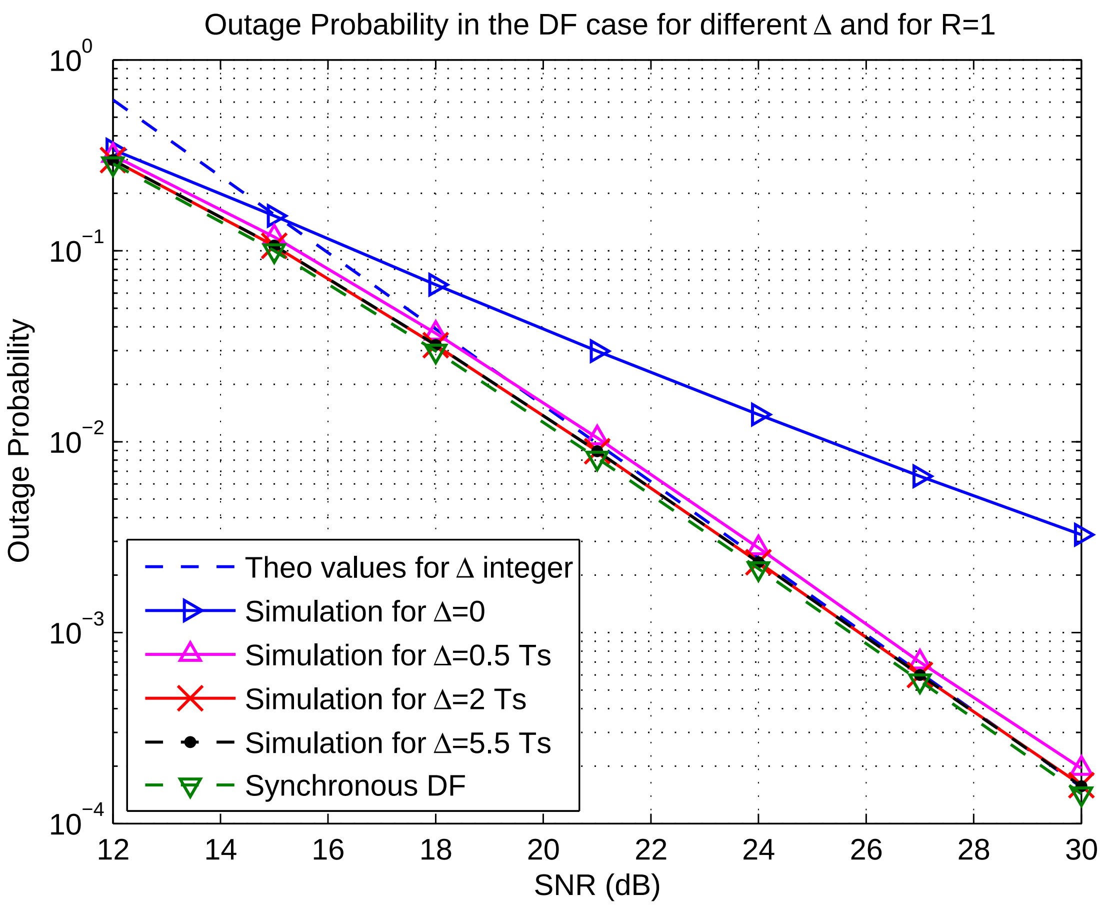

Fig. 2. Outage probability in the DF protocol case for different relative delay Δ and $R = 1$.

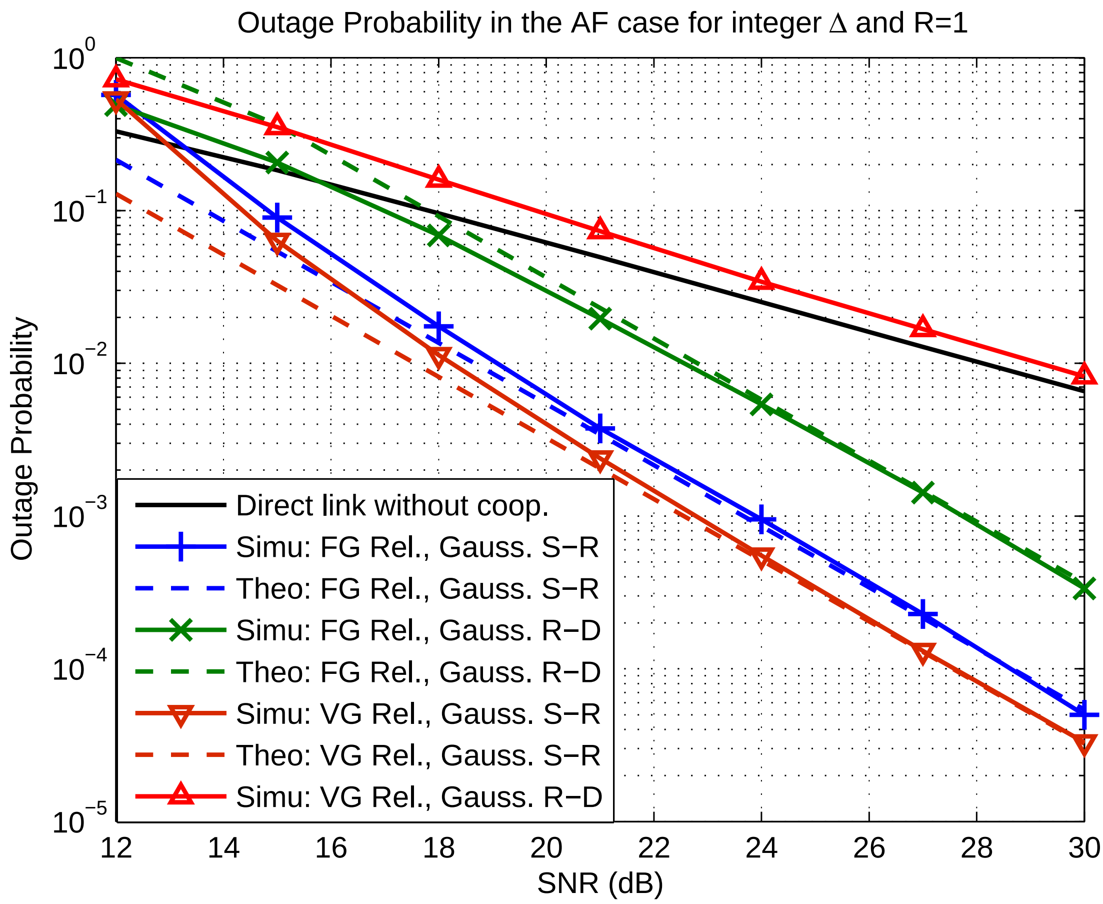

Fig. 3. Outage probability in the AF case of the particular cases with Gaussian source-relay and relay-destination channels for fixed and variable gain relays.

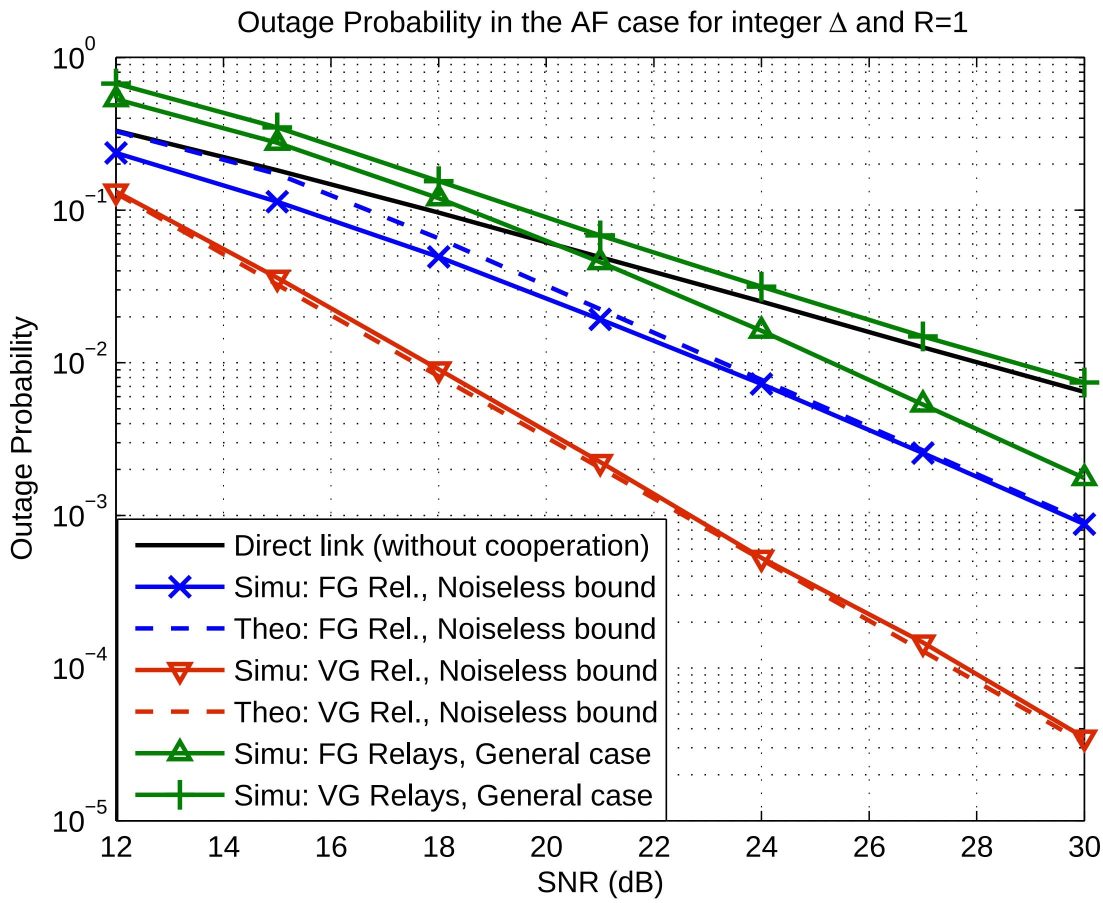

Fig. 4. Outage probability in the AF case of the noiseless relay bound and the general case for fixed and variable gain relays.

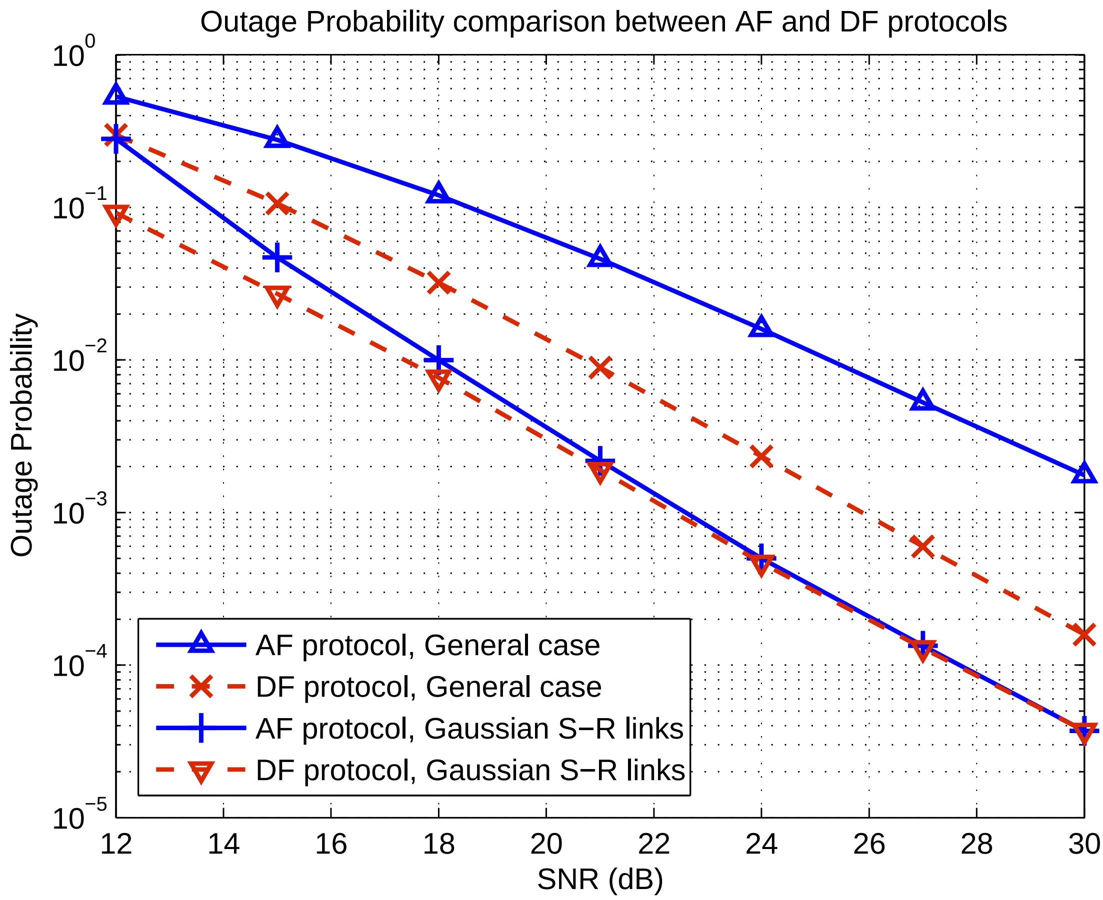

Fig. 5. Outage probability comparison between the DF protocol and the AF protocol for fixed gain relays in the general case and the Gaussian source-relay links case.

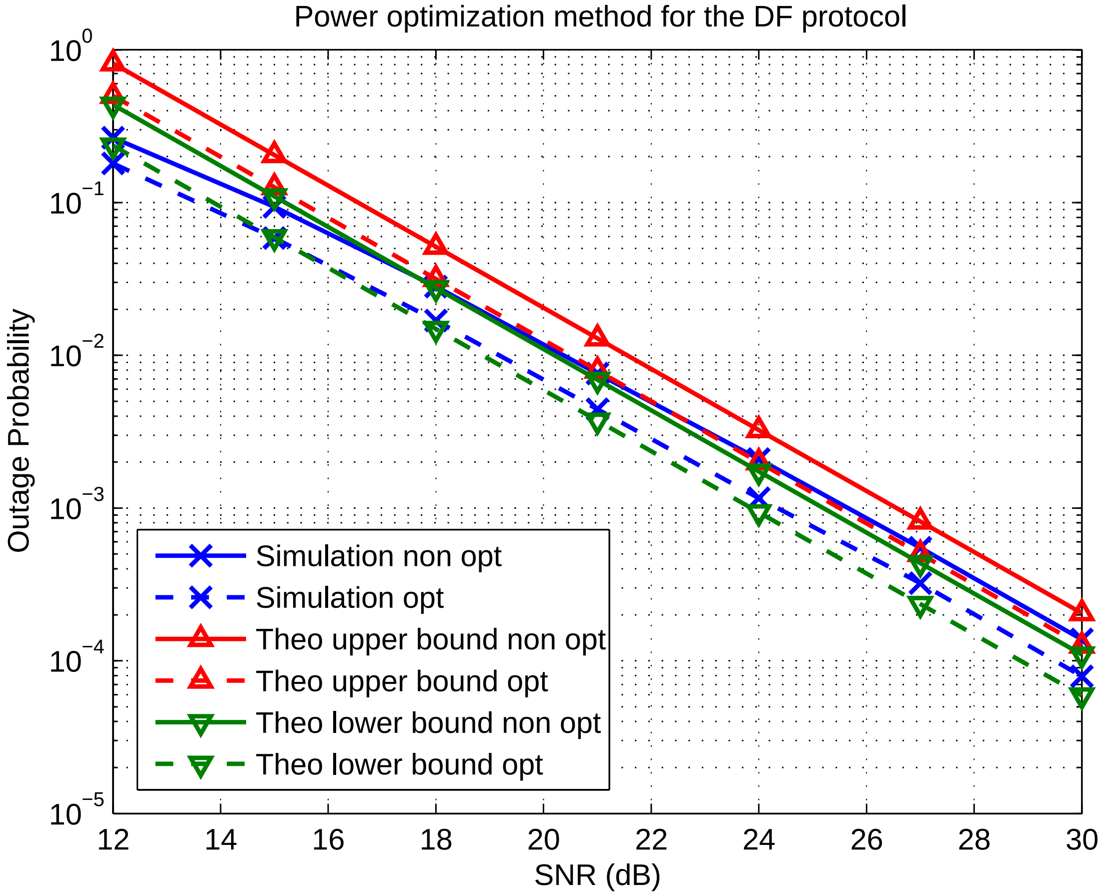

Fig. 6. Outage probability for both equal and optimized power distributions with $\Delta = 1.8T_s$ and $R = 1$.

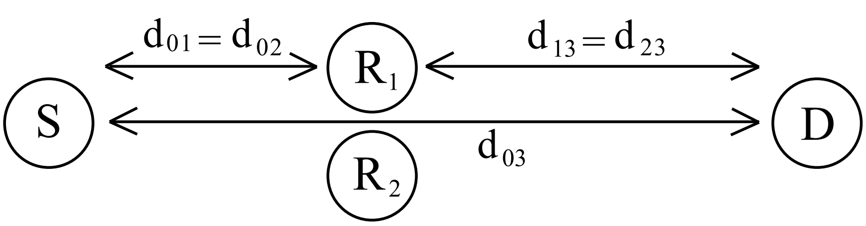

Fig. 7. Network architecture for illustrating power optimization.

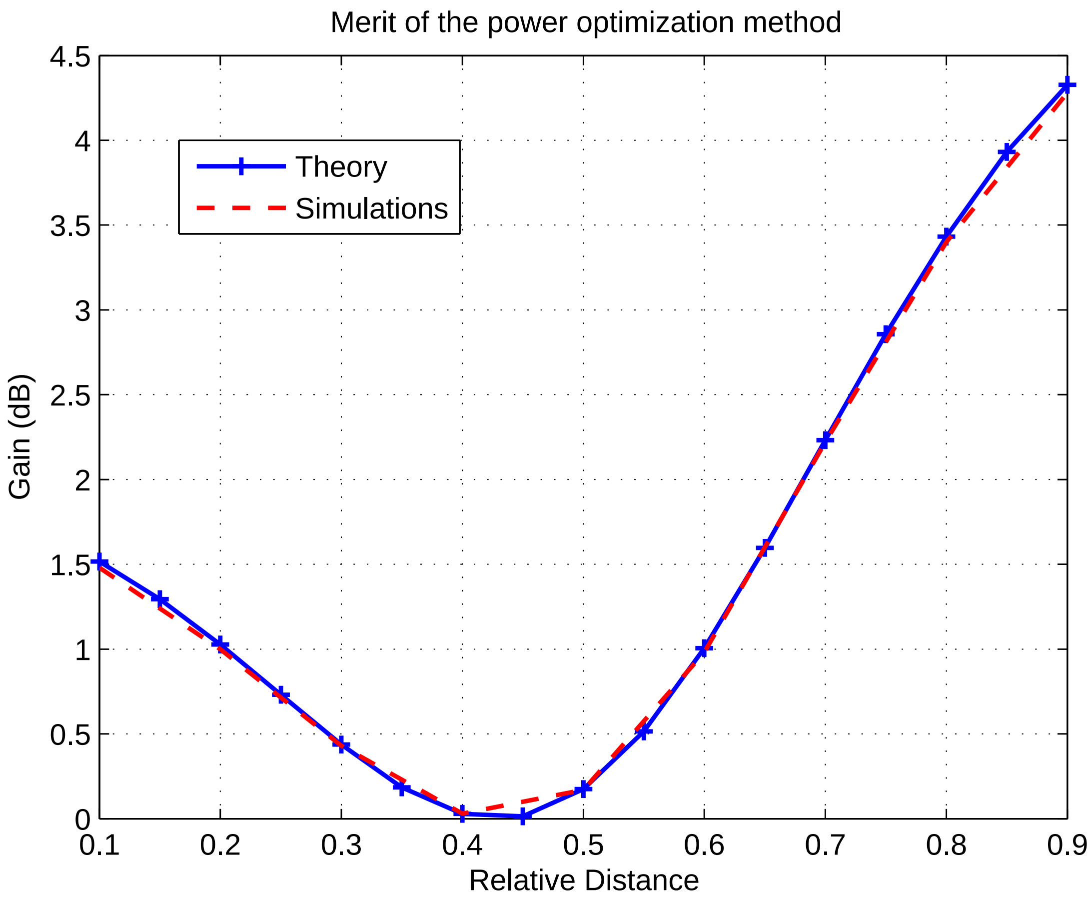

Fig. 8. SNR gain (left) and β_0 values (right) after optimization *vs* Source-Relays relative distance.