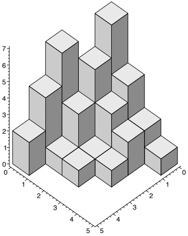
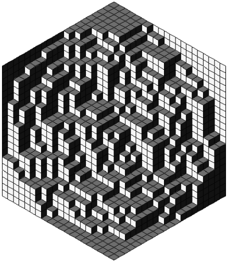
Random sampling of plane partitions

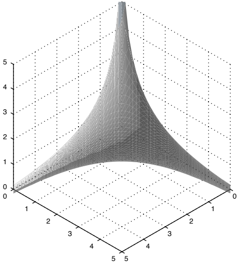
O. Bodini É. Fusy C. Pivoteau

Laboratoire d'Informatique de Paris 6 (LIP6)

INRIA Rocquencourt





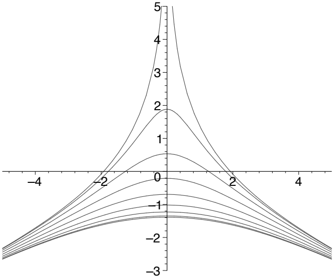


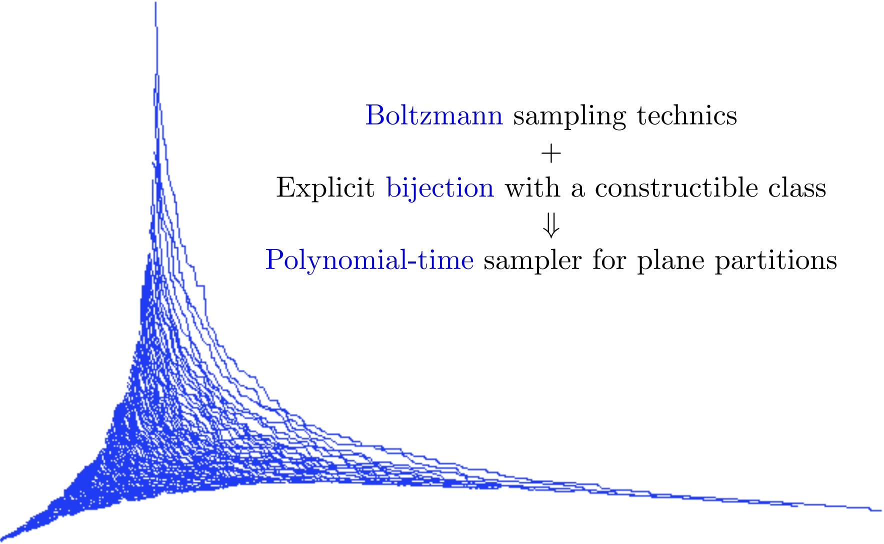
Context

- Young tableaux : natural generalization of integer partitions in 3D,
- huge literature, e.g. the Alternating Sign Matrix Conjecture (Zeilberger 1995),
- Mac Mahon : beautiful (and simple) generating function ($\sim$ 1912)
- for long, no bijective proof,
- Krattenthaler, 1999, proof based on interpretation the hook-length formula,
- sampling of plane partitions in a box $a \times b \times c$:
  $\rightarrow$ hexagon tilings by rhombi,
- 2002 : Pak's bijection for general planes partitions,
- 2004 : Boltzmann sampling
- today : efficient samplers for some classes of plane partitions.

Motivations

- mathematics,
- statistical physics,
- random sampling according to a natural parameter (volume),
- very large object → observation of limit properties,
- in particular : limit shape
  - Cerf and Kenyon,
  - Okounkov and Reshetikhin
- phenomena such as frozen boundaries,
- ...





Boltzmann sampling technics

$+$

Explicit bijection with a constructible class

$\Downarrow$

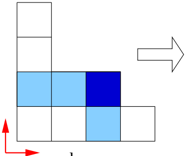
Polynomial-time sampler for plane partitions

Plan of the talk

1. **Pak's bijection**
2. **Boltzmann sampler**
3. **Analysis of Complexity**

Planes partitions

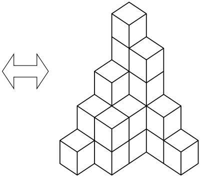
- λ : Integer partition $\simeq$ Shape of plane partition
  e.g. : $\lambda = \{4, 3, 1, 1\}$.
- $h(i, j)$: hook length of the cell (i, j)
- Plane partitions of shape λ ($\mathcal{P}$)
  - λ filled with integers > 0, decreasing in both dimensions
  - matrix filled with integers ≥ 0, decreasing in both dimensions
- Reverse plane partition of shape λ ($\mathcal{RP}$)
  λ filled with integers ≥ 0 that are increasing in both dimensions
- Size of a plane partition : sum of the entries



shape

1	0	0	0
2	0	0	0
4	2	2	0
5	3	2	1

plane partition



3D

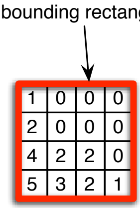
3			
2	4	6	
0	3	5	
0	1	2	4
0	0	2	3

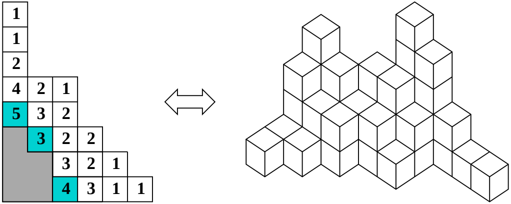
reverse plane partition

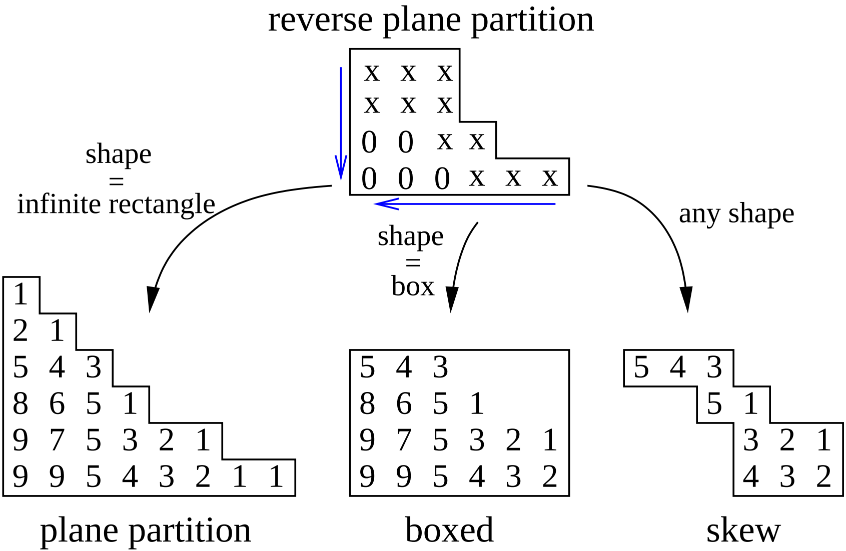
Boxed and skew planes partitions

- Bounding rectangle of a plane partition
  the smallest rectangle containing all the non-zero cells
- $(a \times b)$-boxed plane partitions ($\mathcal{P}_{a,b}$)
  the size of the bounding rectangle is at most $(a \times b)$
- Skew plane partitions ($\mathcal{S}$)
  plane partition of shape λ/μ, where λ, μ are integer partitions and $\lambda \supset \mu$
- Corner of a skew plane partition

$$\mathcal{S} \equiv \mathcal{RP}$$






reverse plane partition
x x x
x x x
0 0 x x
0 0 0 x x x
shape = infinite rectangle
shape = box
any shape
1
2 1
5 4 3
8 6 5 1
9 7 5 3 2 1
9 9 5 4 3 2 1 1
5 4 3
8 6 5 1
9 7 5 3 2 1
9 9 5 4 3 2
5 4 3
5 1
3 2 1
4 3 2
plane partition
boxed
skew

Counting plane partitions

Hook content formula :

$$\sum_{A \in \mathcal{RP}(\lambda)} z^{|A|} = \prod_{(i,j)\in[\lambda]} \frac{1}{1 - z^{h(i,j)}}$$

Set λ to be an infinite rectangle :

$$\prod_{i,j \geq 0} \frac{1}{1 - z^{i+j+1}}$$

Generating function of plane partitions (Mac Mahon, 1912) :

$$P(z) = \prod_{r \geq 1} (1 - z^r)^{-r}$$

- combinatorial isomorphisms with constructible classes (symbolic methods)

$$\mathcal{P} \simeq \mathcal{M} \ , \quad \mathcal{P}_{a,b} \simeq \mathcal{M}_{a,b} \quad \text{and} \quad \mathcal{S}_D \simeq \mathcal{M}_D$$

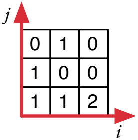
- non-trivial bijection, for long, non constructive proof...

Isomorphic classes

$$\prod_{i,j\geq 0} \frac{1}{1-z^{i+j+1}} = \prod_{i,j\geq 0} \textsc{Seq}(\mathcal{Z}\times\mathcal{Z}^i\times\mathcal{Z}^j) = \textsc{MSet}(\mathcal{Z}\times\textsc{Seq}(\mathcal{Z})^2)$$

- $\mathcal{M} = \textsc{MSet}(\mathbb{N}^2)$ $\sim$ multiset of pairs of integers
  - $\rightarrow$ example : $\{(0,0),(1,0),(2,0),(2,0),(0,1),(1,2)\}$, size $=15$
  - $\rightarrow$ size of (i,j) : $(i+j+1)$
- Diagram of an element $\in \mathcal{M}$

{ (0,0), (1,0), (2,0), (2,0), (0,1), (1,2) } ⇨

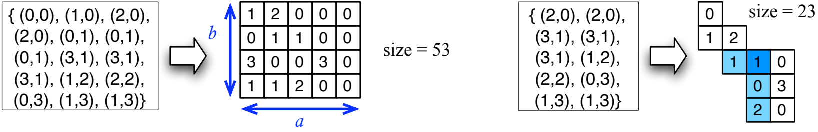


size $= 15$

$|D| = \sum_{i,j} m_{i,j}(i+j+1)$

$\rightarrow$ sum of the hook lengths weighted by the values of the cells.

- $$\mathcal{M}_{a,b} = \text{MSet}(\mathcal{Z} \times \text{Seq}_{<a}(\mathcal{Z}) \times \text{Seq}_{<b}(\mathcal{Z})) = \prod_{\substack{0 \le i < a \\ 0 \le j < b}} \text{Seq}(\mathcal{Z} \times \mathcal{Z}^i \times \mathcal{Z}^j) \sim \text{MSet}(\mathbb{N}_{<a} \times \mathbb{N}_{<b})$$
- $$\mathcal{M}_D = \prod_{(i,j) \in D} \text{Seq}(\mathcal{Z} \times \mathcal{Z}^{i-\ell(i)} \times \mathcal{Z}^{j-d(j)}) = \prod_{(i,j) \in D} \text{Seq}(\mathcal{Z}^{h(i,j)})$$
- Diagrams

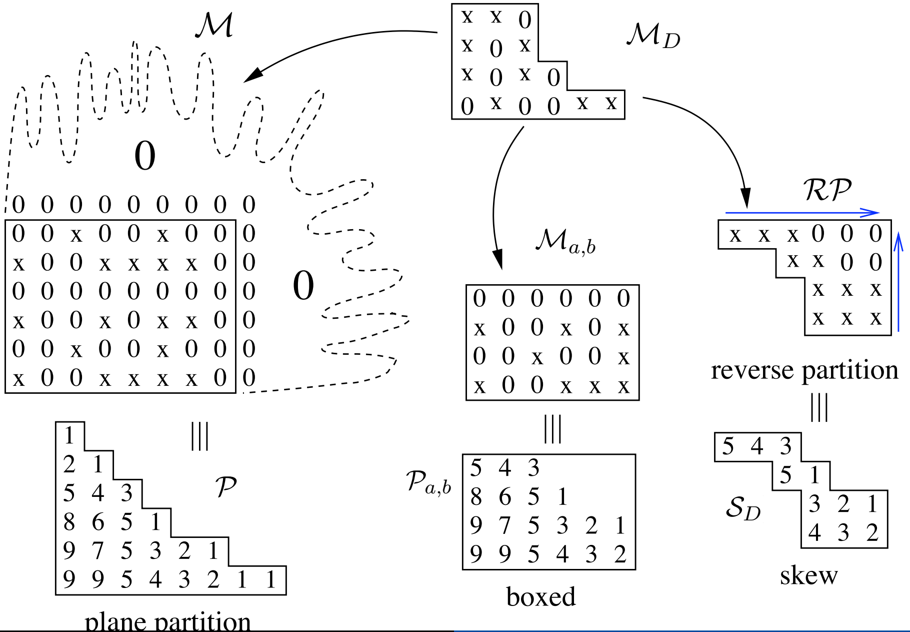


- Hook length of $(i,j) \in D : h(i,j) = (i - \ell(i)) + (j - d(j)) + 1$

  $\ell(i) \leftarrow$ min. abscissa such that $(\ell(i), j) \in D$

  $d(j) \leftarrow$ min. ordinate such that $(i, d(j)) \in D$

Summary


$\mathcal{M}$
$\mathcal{M}_D$
$\mathcal{M}_{a,b}$
$\mathcal{RP}$
0
0
reverse partition
|||
$\mathcal{P}$
$\mathcal{P}_{a,b}$
$\mathcal{S}_D$
skew
boxed
plane partition

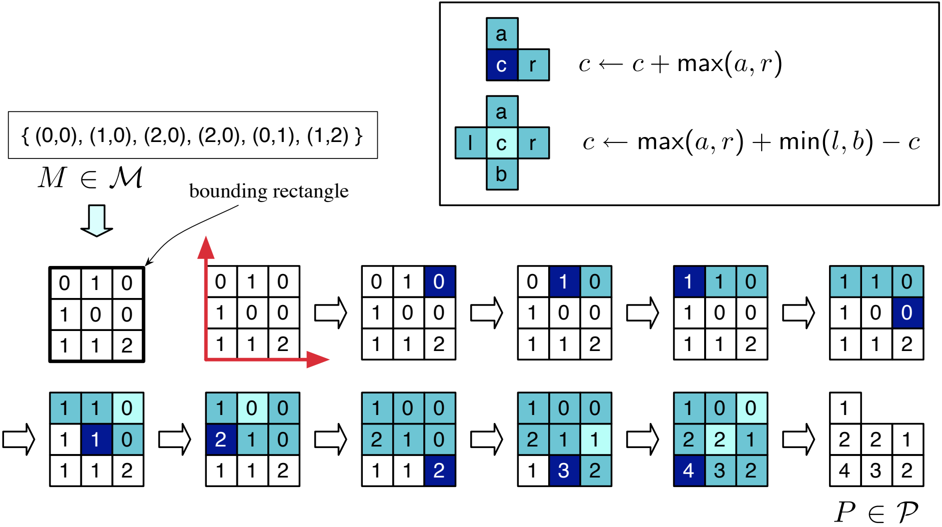
Pak's bijection

Pak's bijection – principles

- sequential update of the corners of the multiset M
- at each step, the current plane partition (of shape λ) correspond to the restriction of M to λ
- prop. 1 : for any corner, the value of the cell, in the plane partition = the maximum value of a monotone path, in the multiset.
- prop. 2 : for any extreme cell, diagonal sum, in the plane partition = rectangular sum, in the multiset.
- order constraint, size constraint
- dynamic programming

simple algorithm, but difficult proof!

Pak's bijection – illustrated example



Application of Pak's algorithm on an example.

Pak's algorithm

Input : a diagram D of a multiset in $\mathcal{M}$.
Output: a plane partition.
Let ℓ be the length and w be the width of D.

```
for i := ℓ − 1 downto 0 do
    for j := w − 1 downto 0 do
        D[i, j] ← D[i, j] + max(D[j + 1, i]), D[i, j + 1]);
        for c := 1 to min(w − 1 − j, ℓ − 1 − i) do
            x ← i + c ; y ← j + c;
            D[x, y] ← max(D[x + 1, y], D[x, y + 1]) ;
                      + min(D[x + 1, y], D[x, y + 1]);
                      −D[x, y];
Return D;
```

Boltzmann sampler

Random sampling under Boltzmann model

- for any constructible class
- approximate size sampling,
- size distribution spread over the whole combinatorial class, but uniform for a sub-class of objects of the same size,
- control parameter,
- automatized sampling : the sampler is compiled from specification automatically,
- very large objects can be sampled.

Model definition

Definition

In the unlabelled case, Boltzmann model assigns to any object $c \in \mathcal{C}$ the following probability :

$$\mathbb{P}_x(c) = \frac{x^{|c|}}{C(x)}$$

A Boltzmann sampler $\Gamma C(x)$ for the class $\mathcal{C}$ is a process that produces objects from $\mathcal{C}$ according to this model.

$\rightarrow$ 2 object of the same size will be drawn with the same probability.

The probability of drawing an object of size N is then :

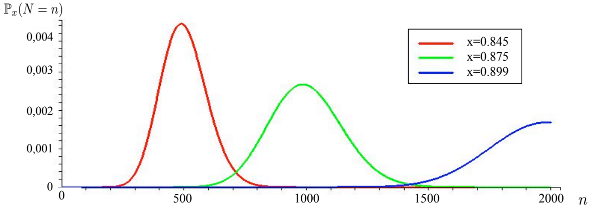
$$\mathbb{P}_x(N = n) = \sum_{|c|=n} \mathbb{P}_x(c) = \frac{C_n x^n}{C(x)}$$

Then, the expected size of an object drawn by a generator with parameter x is :

$$\mathbb{E}_x(N) = x\frac{C'(x)}{C(x)}$$

Approximate and exact-size samplers

- Free samplers : produce objects with randomly varying sizes !
- Tuned samplers : choose x so that expected size is n.
- Run the targeted sampler until the output size is in the desired range (rejection).
- Size distribution of free sampler determines complexity.



Unions, products, sequences

Disjoint unions

Boltzmann sampler ΓC for $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$:

With probability $\frac{A(x)}{C(x)}$ do $\Gamma A(x)$ else do $\Gamma B(x)$ $\longrightarrow$ Bernoulli.

Products

Boltzmann sampler ΓC for $\mathcal{C} = \mathcal{A} \times \mathcal{B}$:

Generate a pair $\langle\ \Gamma A(x)\ ,\ \Gamma B(x)\ \rangle$ $\longrightarrow$ independent calls.

Sequences

Boltzmann sampler ΓC for $\mathcal{C} = \textsc{Seq}(\mathcal{A})$:

Generate k according to a geometric law of parameter $A(x)$

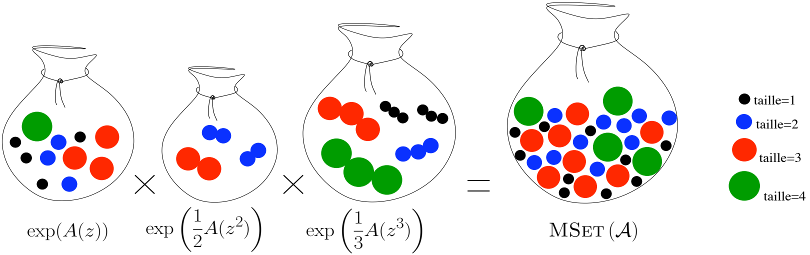
Generate a k-tuple $\langle\ \Gamma A(x)\ , \dots ,\ \Gamma A(x)\ \rangle$ $\longrightarrow$ independent calls.

Remark : $A(x)$, $B(x)$, and $C(x)$ is given by an ***oracle***.

Generating multisets

$$\mathcal{C} = \text{MSet}(\mathcal{A}) \cong \prod_{\gamma \in \mathcal{A}} \text{Seq}(\gamma) \quad \Rightarrow \quad C(z) = \prod_{\gamma \in \mathcal{A}} (1 - z^{|\gamma|})^{-1}$$

$$C(z) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} A(z^k)\right) = \prod_{k=1}^{\infty} \exp\left(\frac{1}{k} A(z^k)\right)$$



Sampling an object of $\mathcal{M}$

Algorithm $\Gamma M(x)$

M is the diagram of the multiset to be generated

- Draw m, the max. index of a subset, depending on x ;
- For each index k of a subset until $m-1$
  - Draw the number p of elements to sample, according to a Poisson law of parameter $\frac{x^k}{k(1-x^k)^2}$.
  - Perform p calls to the sampler for $\mathcal{Z} \times \textsc{Seq}(\mathcal{Z})^2$ with parameter x^k, and each time, add k copies of the result to the multiset.

    Repeat p times :

    $i \leftarrow \mathrm{Geom}(x^k)$;
    $j \leftarrow \mathrm{Geom}(x^k)$;
    $M[i,j] \leftarrow M[i,j] + k$
- for index m, draw the number p of elements to generate, according to a non zero Poisson law.

Sampling $\mathcal{M}_{a,b}$ an $\mathcal{M}_D$

$\Gamma M_{a,b}(x)$ [Boltzmann sampler for $\mathcal{M}_{a,b}$]

M is the diagram of the multiset to be generated
for $i \leftarrow 0$ **to** $a-1$ **do**
    for $j \leftarrow 0$ **to** $b-1$ **do**
        $M[i,j] \leftarrow \mathrm{Geom}(x^{i+j+1})$;
return M;

$\Gamma S_D(x)$ [Boltzmann sampler for $\mathcal{M}_D$]

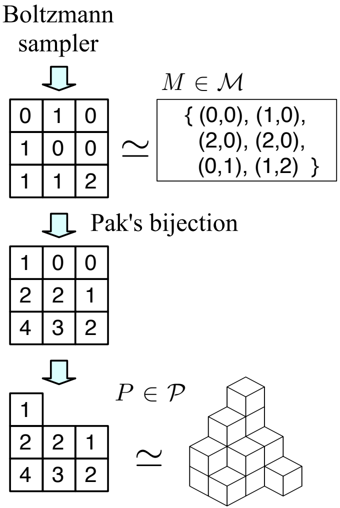
M is the diagram of the multiset to be generated
for $(i,j) \in D$ **do**
    $M[i,j] \leftarrow \mathrm{Geom}(x^{i+j+1})$;
return M;

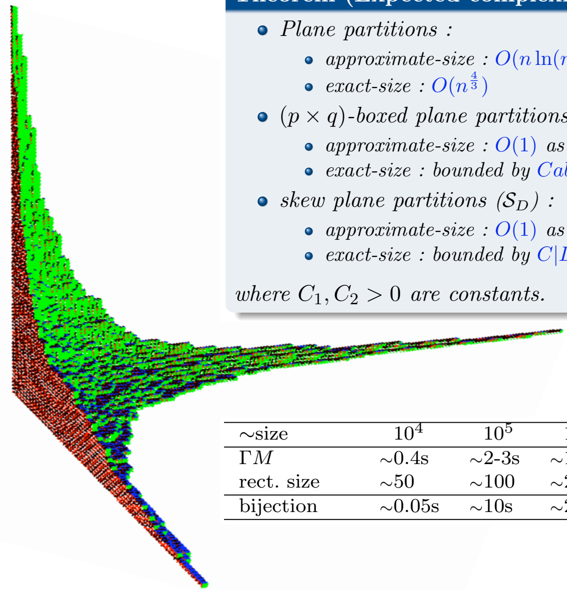
! the free Boltzmann samplers operate in linear time in the size of the bounding rectangle of the diagram produced.

Summary

- Targeted Boltzmann sampler for
  - $\mathcal{M} \to$ plane partitions
  - $\mathcal{M}_{a,b} \to$ boxed plane partitions
  - $\mathcal{S}_D \to$ skew planes partitions

  Output : a *diagram* D.
- Rejection
- Pak's algorithm transforms D into a plane partition.
- Size of the output plane partition $=$ size of the original diagram.





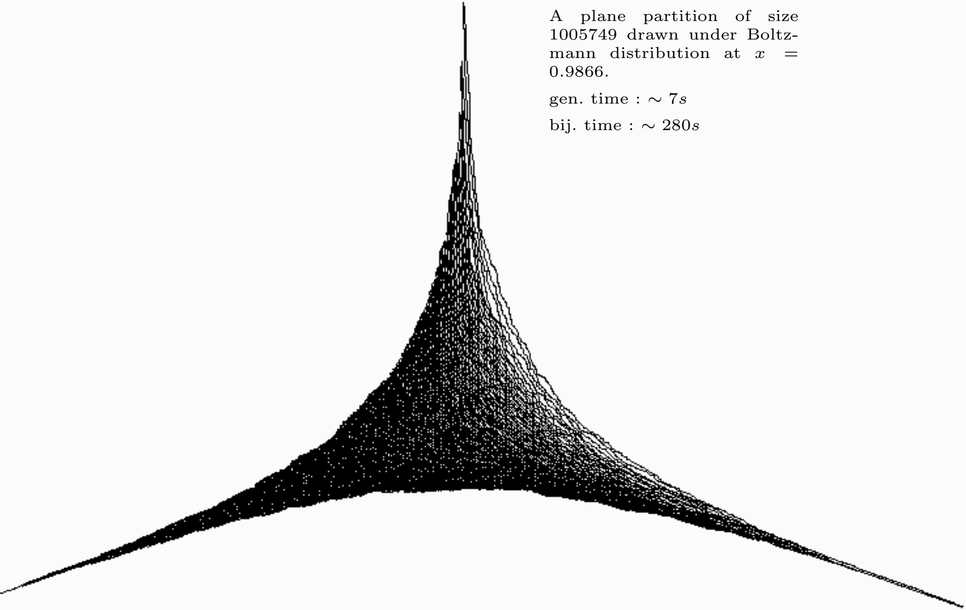
Theorem (Expected complexity)

- *Plane partitions :*
  - *approximate-size :* $O(n \ln(n)^3)$
  - *exact-size :* $O(n^{\frac{4}{3}})$
- *$(p \times q)$-boxed plane partitions (for fixed a, b) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $Cab.n$
- *skew plane partitions ($\mathcal{S}_D$) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $C|D|.n$

where $C_1, C_2 > 0$ *are constants.*

~size	10^4	10^5	10^6	10^7
ΓM	~0.4s	~2-3s	~10s	~60s
rect. size	~50	~100	~200-300	~600-800
bijection	~0.05s	~10s	~20s	~250-300s

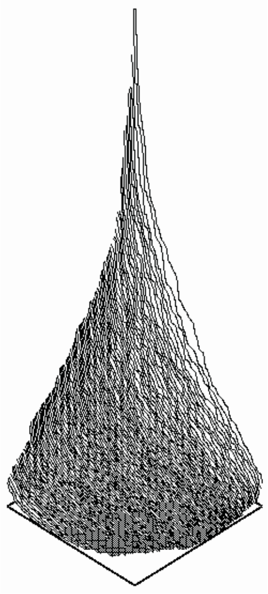
Results – 2



A plane partition of size 1005749 drawn under Boltzmann distribution at $x = 0.9866$.

gen. time : $\sim 7s$

bij. time : $\sim 280s$



← A (100 × 100)-boxed plane partition of size 999400 drawn under Boltzmann distribution at $x = 0.9931$.

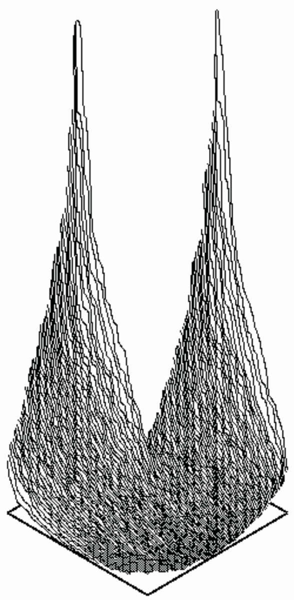
gen. time : $\sim 5s$

bij. time : $\sim 0.7s$

→ A skew plane partition of size 1005532 on the index-domain :
$[0..99]\times[0..99]\setminus[0..49]\times[0..49]$, drawn under Boltzmann distribution at $x = 0.9942$.

gen. time : $\sim 4s$.

bij. time : $\sim 0.35s$.



Analysis of Complexity

Theorem (Expected complexity)

- *Plane partitions :*
  - *approximate-size :* $O(n \ln(n)^3)$
  - *exact-size :* $O(n^{\frac{4}{3}})$
- *$(p \times q)$-boxed plane partitions (for fixed a, b) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $Cab.n$
- *skew plane partitions ($\mathcal{S}_D$) :*
  - *approximate-size :* $O(1)$ *as* $n \to \infty$
  - *exact-size : bounded by* $C|D|.n$

where $C_1, C_2 > 0$ *are constants.*

General scheme

Generation of a plane partition of size n (resp. $\sim n$), with a targeted sampler, i.e., with a parameter tuned such that $\mathbb{E}(N_x) = n$.

mean cost

=

cost of one call to ΓM $\times$ expected number of calls

+

cost of Pak's algorithm

1. cost of one call to ΓM : $O(n^{\frac{2}{3}})$
2. expected number of calls to the sampler :
   - approximate size sampler : $O(1)$
   - exact size sampler : $O(n^{\frac{2}{3}})$
3. expected complexity of Pak's algorithm applied to a diagram of size n : $O(n \ln(n)^3)$

Details – free sampler

complexity of the free Boltzmann sampler, as $x \to 1^-$:

$$\Lambda P(x) = \Lambda M(x) + \mathbb{E}_x[\text{PakAlgo}](x)$$

$$\Lambda M(x) = \sum_{i \geq 1} \mathbb{E}\left(\text{Pois}\left(\frac{A(x^i)}{i}\right)\right) \Lambda A(x^i) = \sum_{i \geq 1} \frac{A(x^i)}{i} \Lambda A(x^i)$$

using Mellin transform :

$$\Lambda M(x) = \underset{x \to 1^-}{\mathcal{O}} \left(\frac{1}{(1-x)^2}\right)$$

length of the bounding rectangle of a multiset drawn under Boltzmann model : $\mathcal{O}((1-x)^{-1} \ln((1-x)^{-1}))$ as $x \to 1^-$:

$$\mathbb{E}_x[\text{PakAlgo}](x) = \underset{x \to 1^-}{\mathcal{O}} \left(\frac{1}{(1-x)^3} \ln\left(\frac{1}{1-x}\right)^3\right) = \Lambda P(x)$$

Details – targeted sampler

using Mellin transform :

$$\mathbb{E}(N_x) = \frac{2\zeta(3)}{(1-x)^3} + \underset{x\to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^2}\right)$$
$$\mathbb{V}(N_x) = \frac{6\zeta(3)}{(1-x)^4} + \underset{x\to 1^-}{\mathcal{O}}\left(\frac{1}{(1-x)^3}\right)$$

tuned parameter : $\xi_n := 1 - (2\zeta(3)/n)^{1/3}$

expected complexity of $\Gamma M(\xi_n)$ and Pak's algorithm under the uniform distribution at a *fixed* size n :

$$\Lambda M(\xi_n) = \mathcal{O}(n^{\frac{2}{3}}), \qquad \mathbb{E}_n[\text{Pak}] = \mathcal{O}(n \log(n)^3)$$

probability that the output of $\Gamma P(\xi_n)$ has size n :

- using Chebyshev inequality : $\pi_{n,\epsilon} \underset{n\to\infty}{\longrightarrow} 1$
- using Mellin transform and the saddle-point method :
$$\pi_n \underset{n\to\infty}{\sim} \frac{c}{n^{2/3}}, \text{ with } c \approx 0.1023$$

Details – boxed, skew

sampler for $(a \times b)$-boxed plane partitions :

$$\xi_n^{a,b} := 1 - ab/n$$

$$\pi_{n,\epsilon} \underset{n\to\infty}{\sim} \mathcal{O}(1), \qquad \pi_n \sim \mathcal{O}(n)$$

$\Gamma P_{a,b}(x)$ is of constant complexity $C \cdot a \cdot b$

expected complexity of the approximate-size sampler :

$$\Lambda P_{a,b}(\xi_n)/\pi_{n,\epsilon} \sim C \cdot ab$$

expected complexity of the exact-size sampler :

$$\Lambda P_{a,b}(\xi_n)/\pi_n \sim Cabn$$

Bibliography

- Plane partitions and applications.
  - The low-temperature expansion of the Wulff crystal in the 3D Ising model. R. Cerf, R. Kenyon.
  - Another involution principle-free bijective proof of Stanley's hook-content formula. C. Krattenthaler.
  - Random skew plane partitions and the pearcey process. A. Okounkov, N. Reshetikhin.
  - Partition bijections, a survey. I. Pak.

- Random generation under Boltzmann model
  - Boltzmann samplers for the random generation of combinatorial structures. P. Duchon, P. Flajolet, G. Louchard, G. Schaeffer.
  - Boltzmann sampling of unlabelled structures. P. Flajolet, E. Fusy, C. Pivoteau.

- Pak's bijection
  - Hook length formula and geometric combinatorics. I. Pak.