Scaling limit of random planar maps

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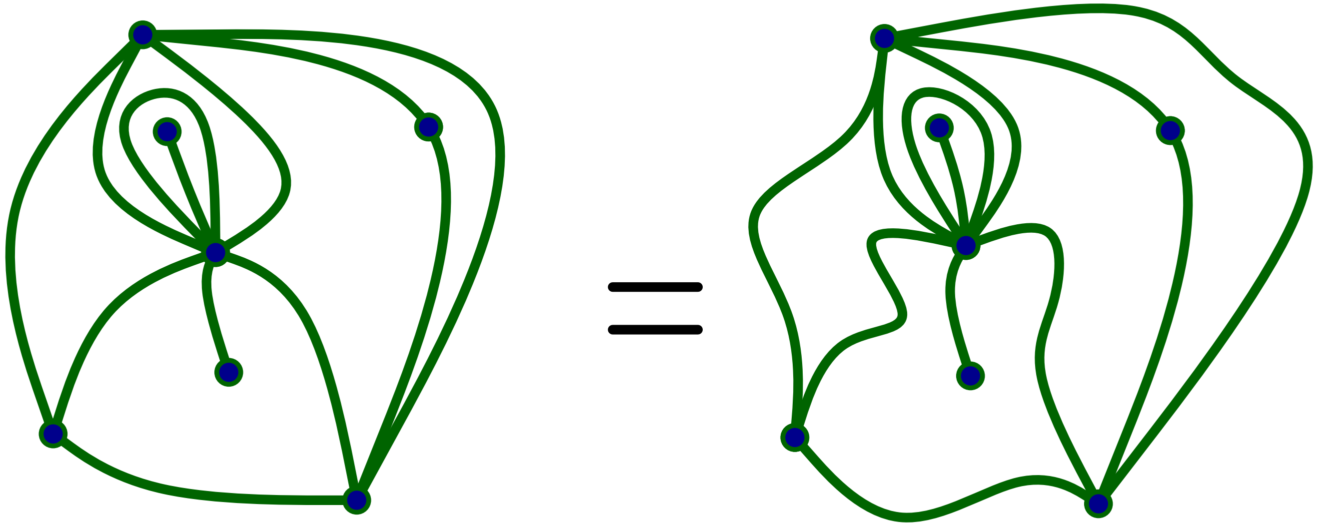
Séminaire Géométrie Algorithmique et Combinatoire,
January 2018

Planar Maps – Definition.

A **planar map** is the proper embedding of a finite connected graph in the 2-dimensional sphere seen up to continuous deformations.



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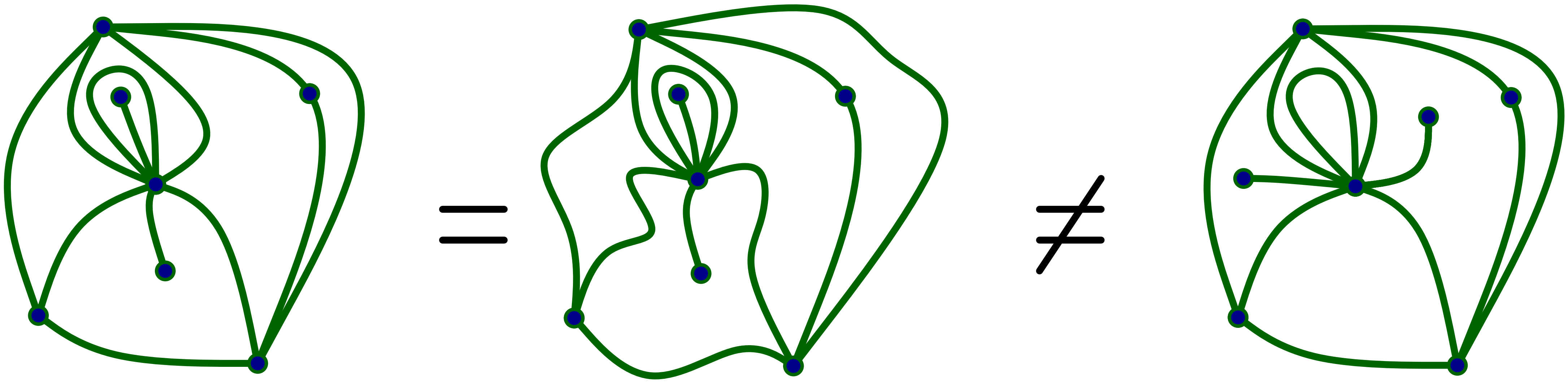

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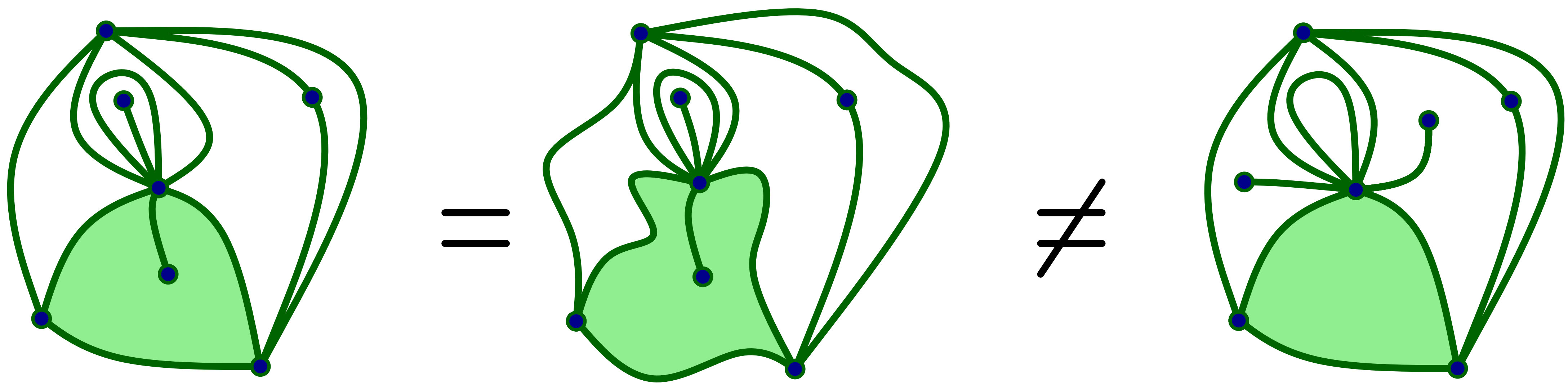

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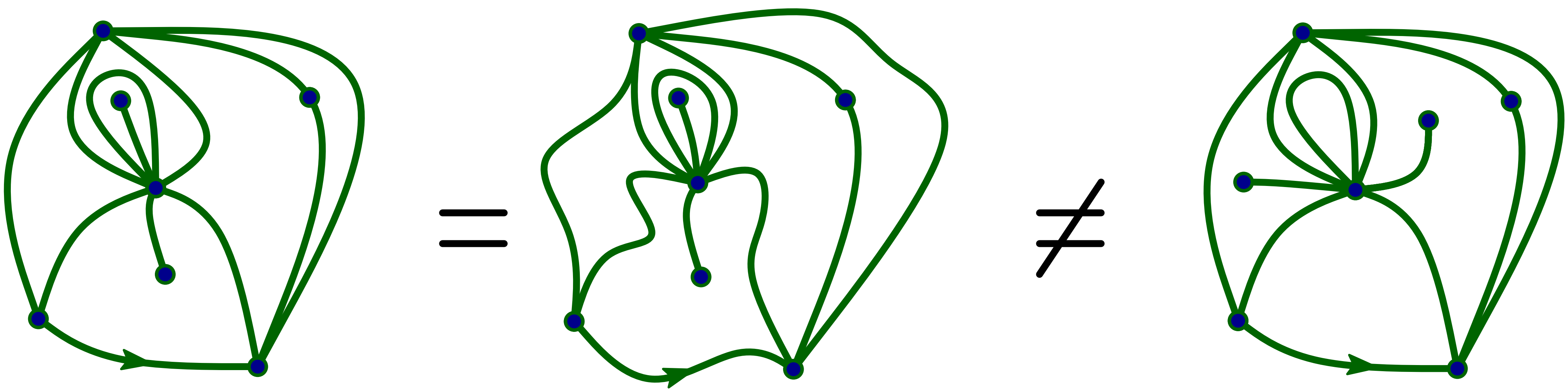

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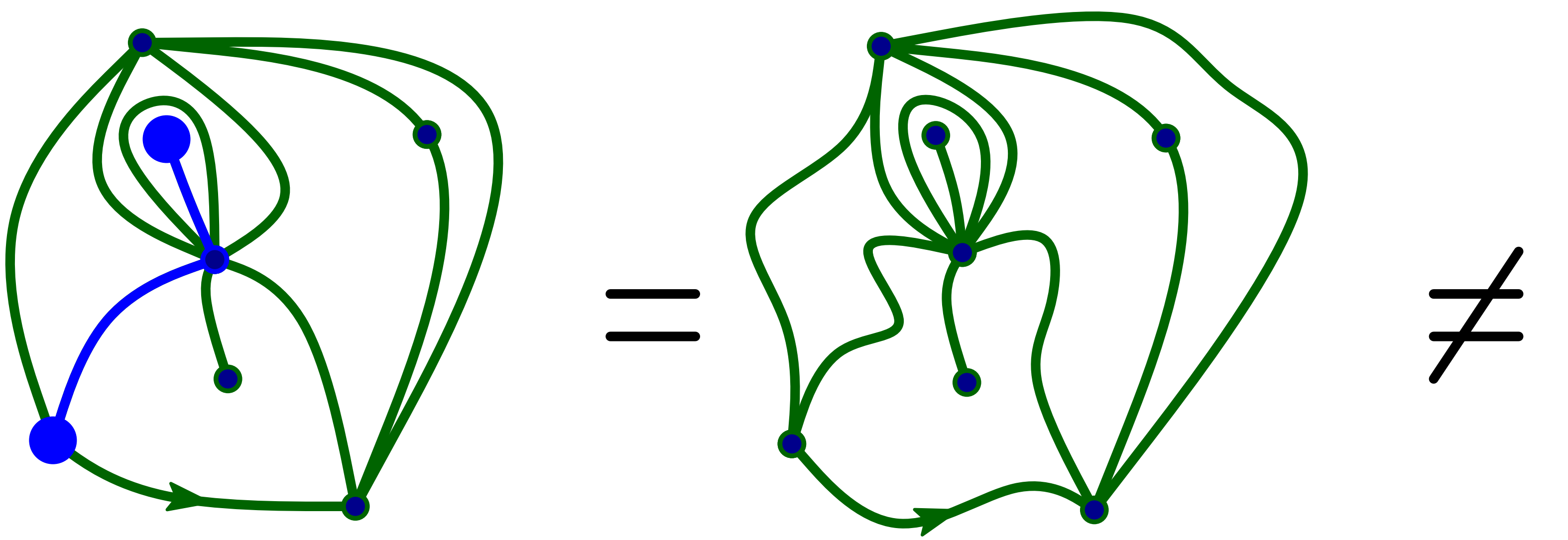

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Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

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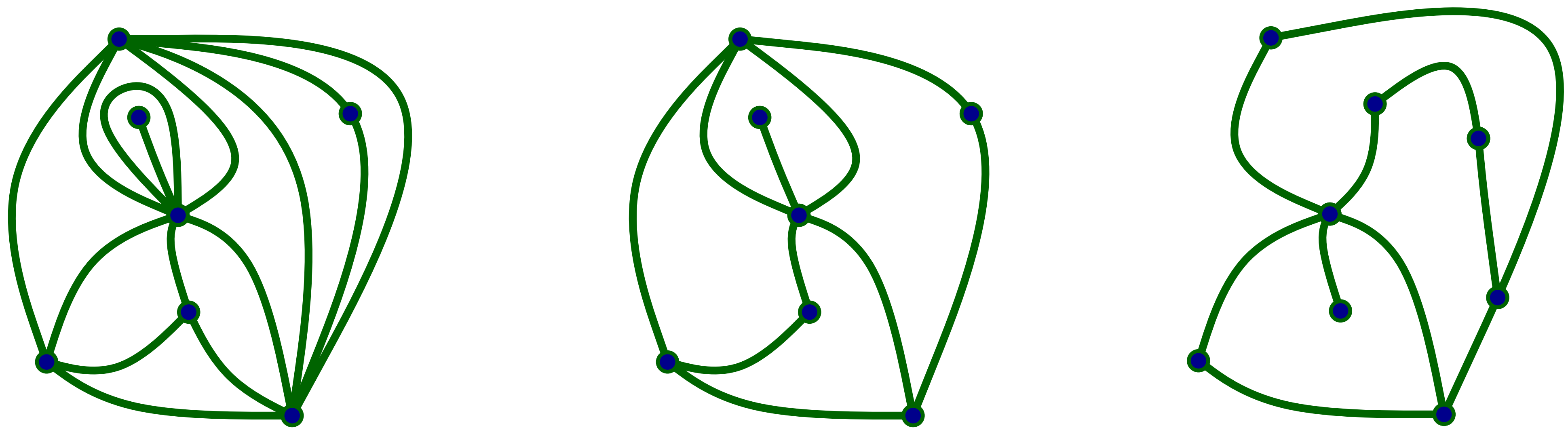
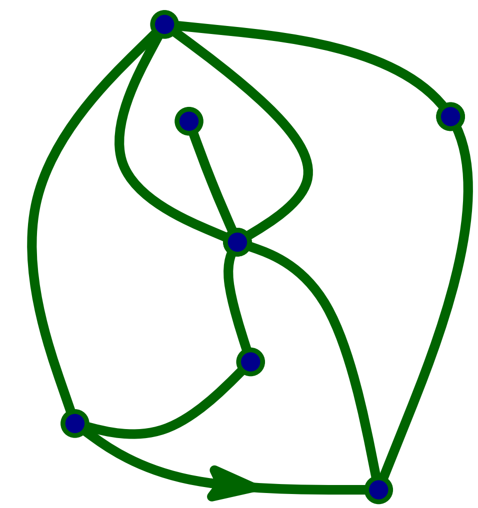


Triangulations, **quadrangulations**, **pentagulations**, **p-angulations** have faces of degree 3, 4, 5, p.

Random quadrangulations



$\mathcal{Q}_n = \{\text{Quadrangulations of size } n\}$
$= n+2$ vertices, n faces, $2n$ edges

Q_n = Random element of $\mathcal{Q}_n$

Simplest model of maps + quadrangulations with n faces are in bijection with general maps with n edges.

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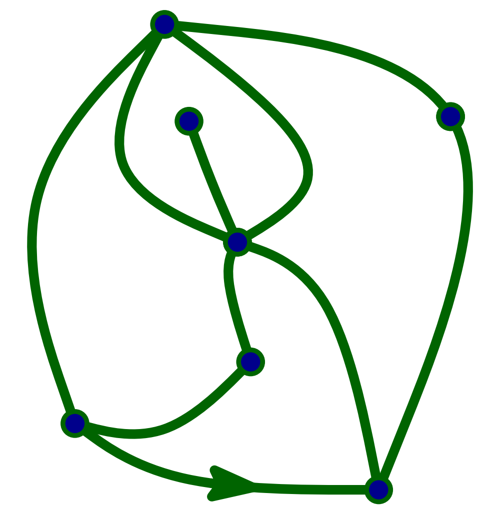

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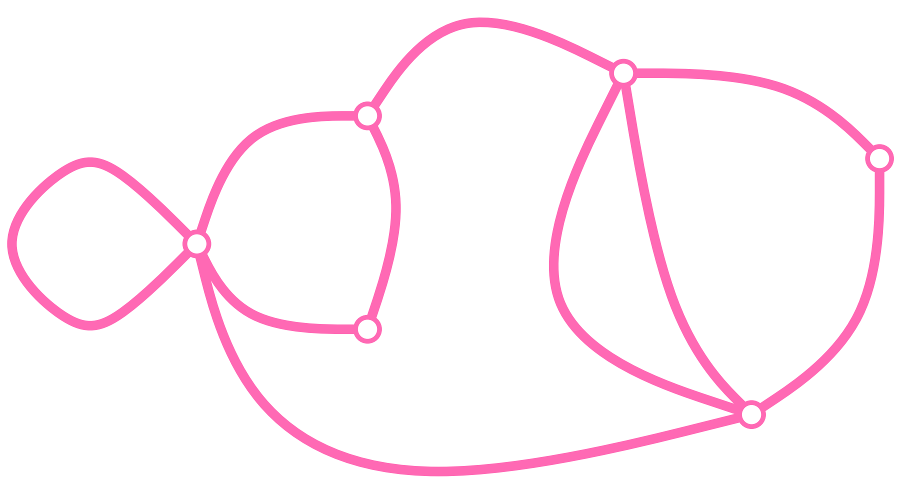

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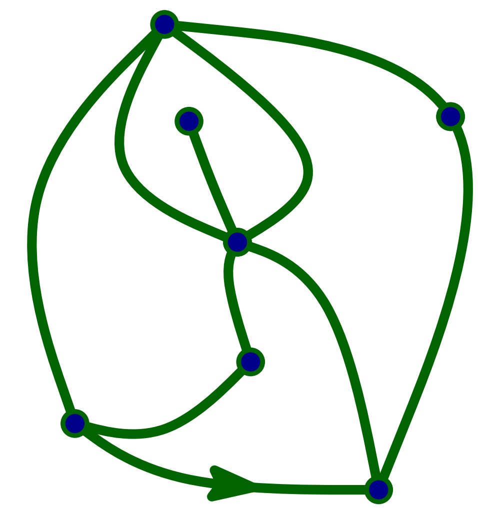

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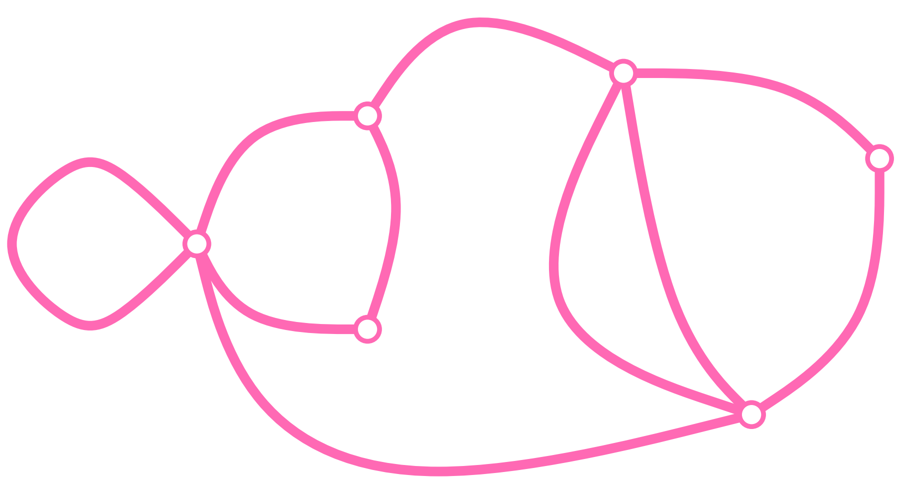

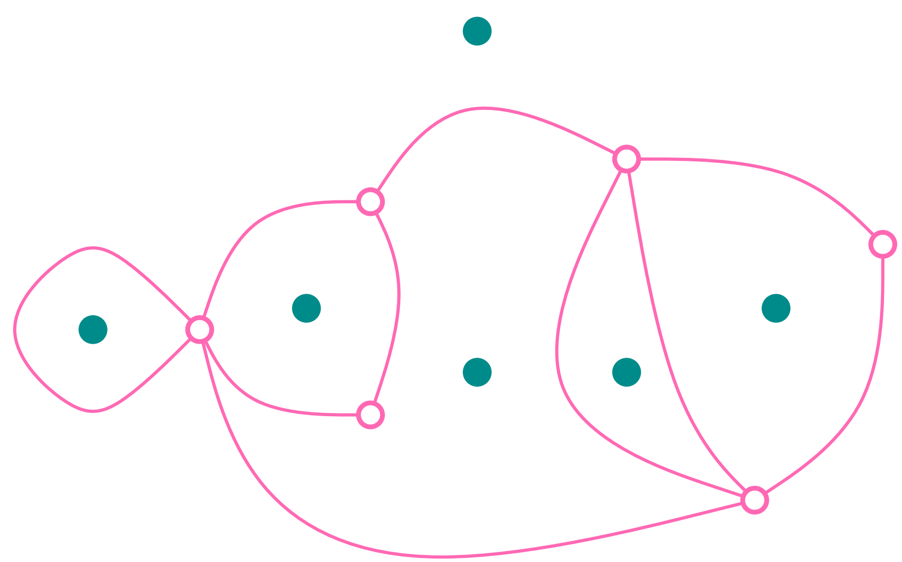

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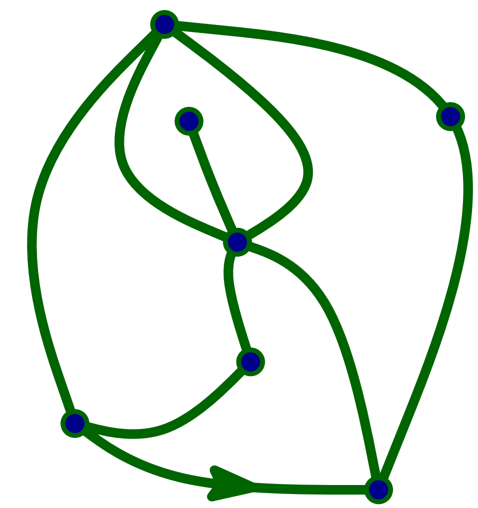

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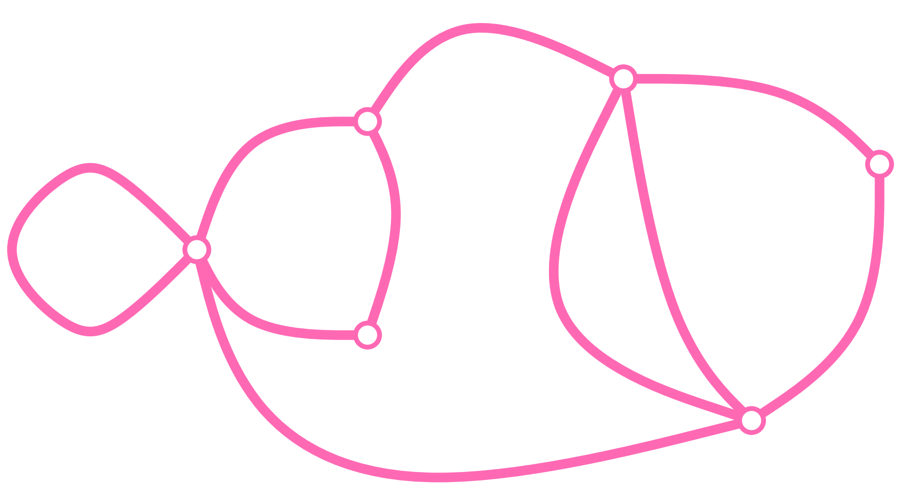

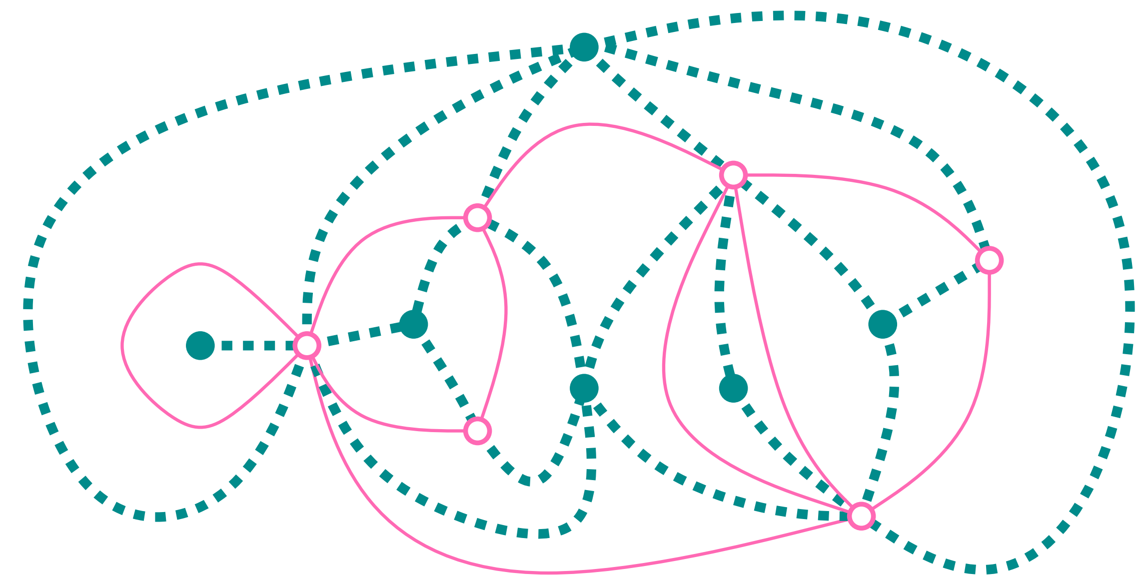

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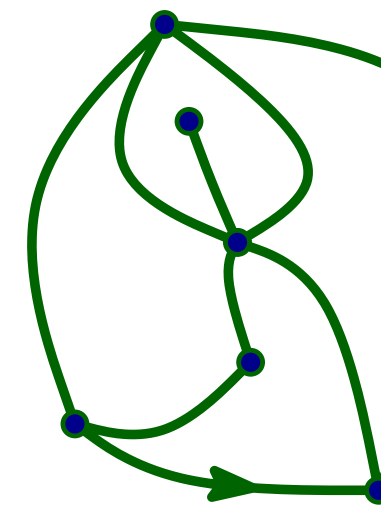

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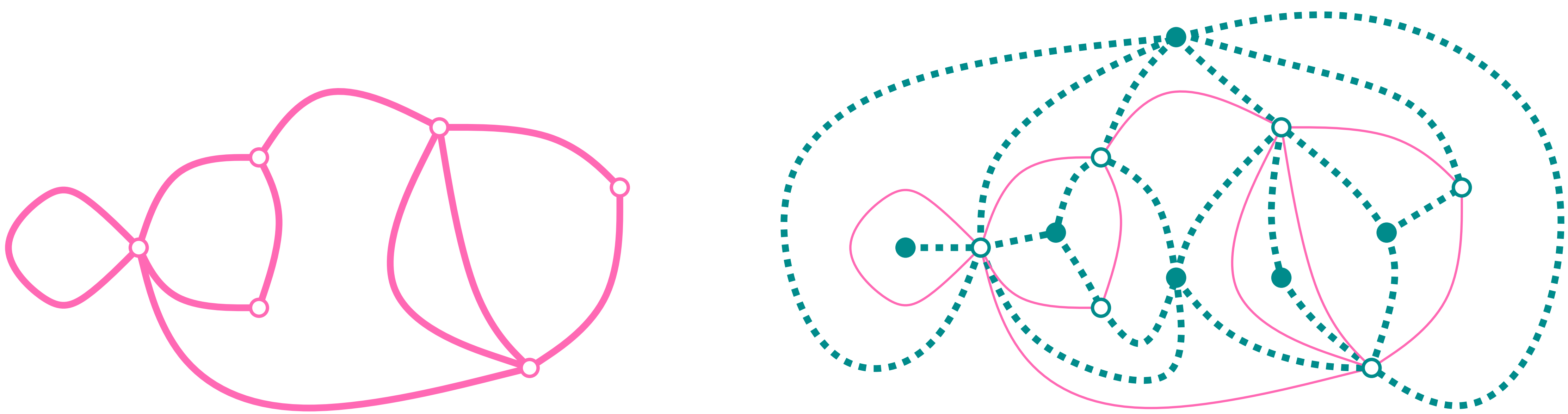

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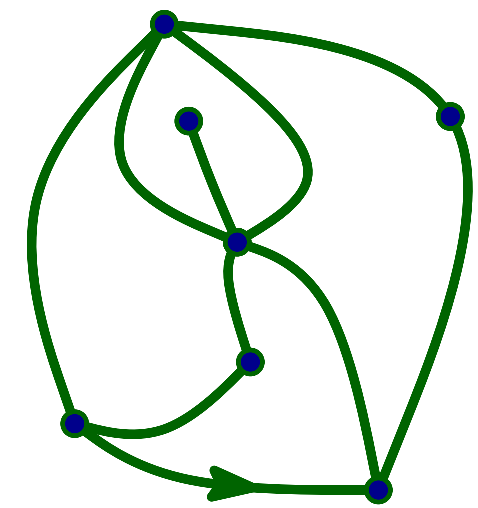

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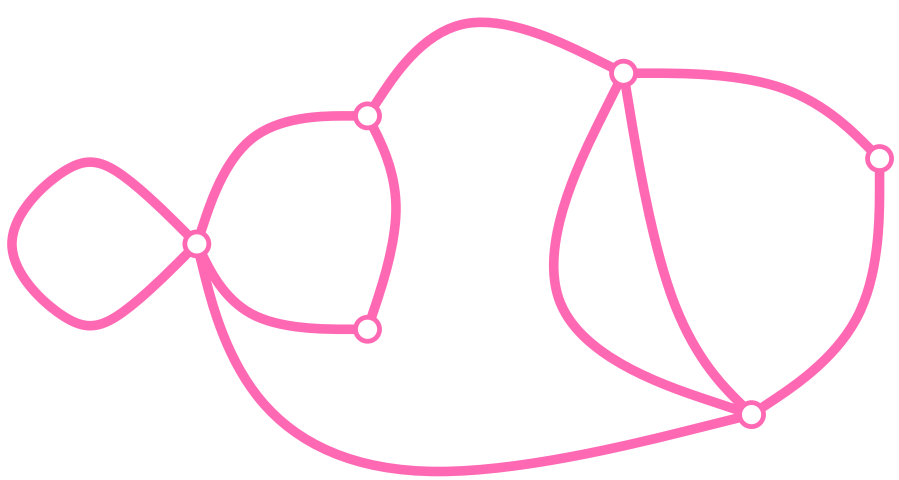

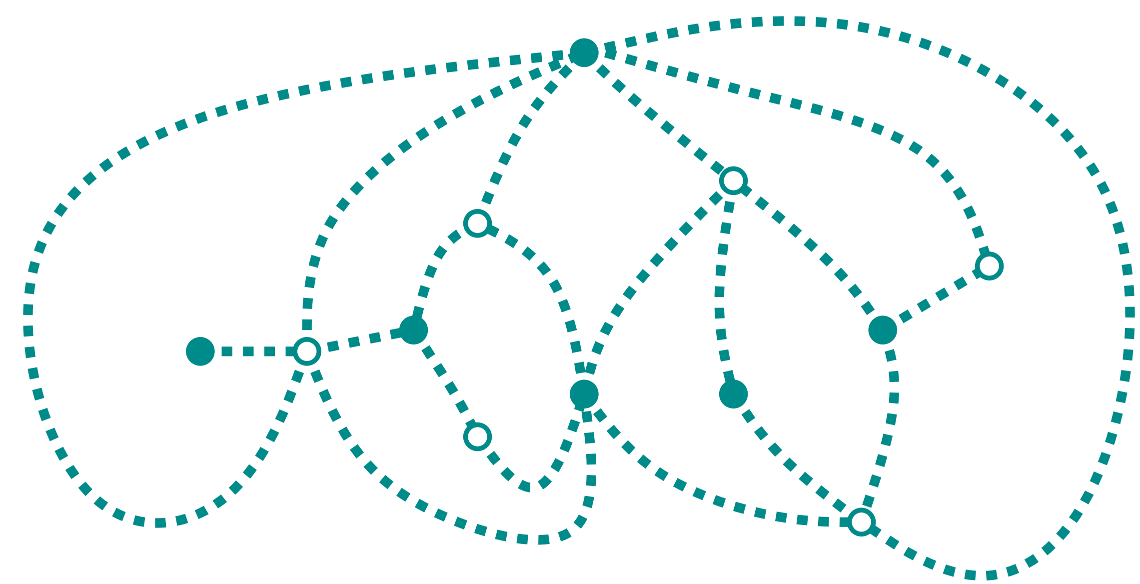

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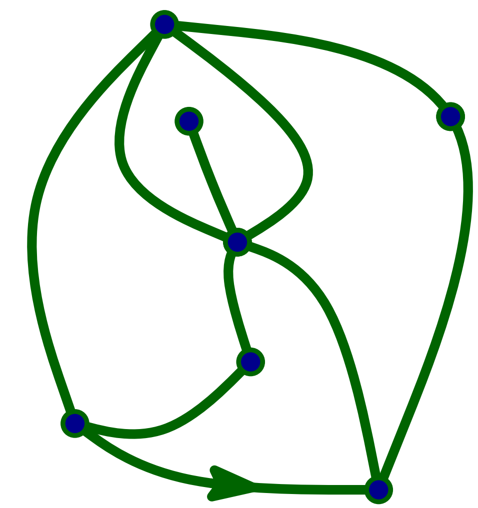

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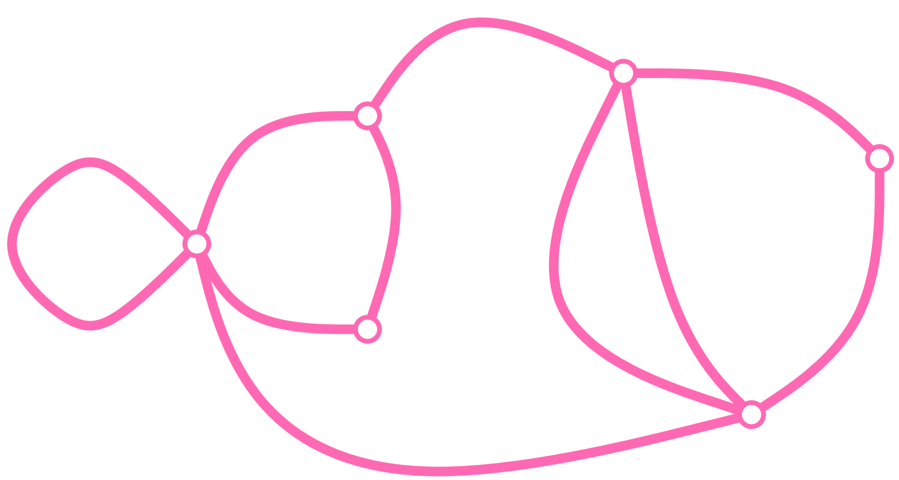

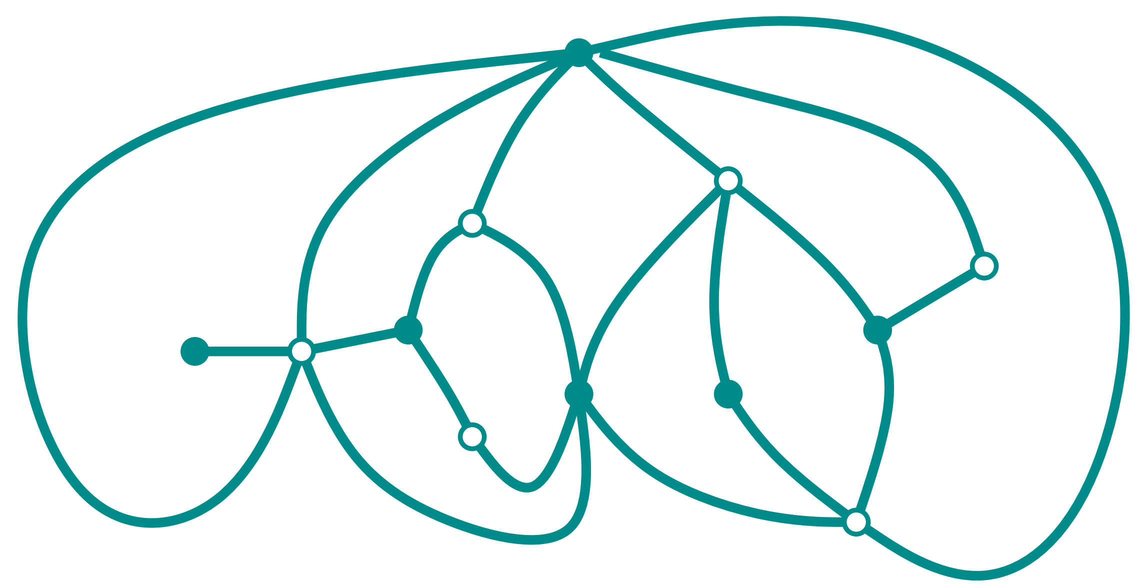

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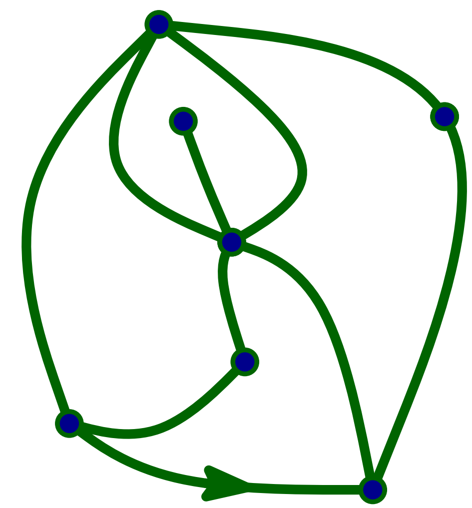

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$(V(Q_n), d_{gr})$ is a random compact metric space

What is the behavior of Q_n when n goes to infinity ?
typical distances?
convergence towards a continuous object ?

Motivations

- Discretization of a continuous surface.

- Construction of a 2-dimensional analogue of the Brownian motion.

Idea : "good" random walks converge (when properly rescaled) to the Brownian motion

Can we have a similar statement in 2 dimensions ? **Brownian map ?**

Universality ? : "good" models of maps converge to the Brownian map ?

- KPZ relations = relation between critical exponents on fixed lattice and on random lattices [Duplantier-Sheffield], [Miller-Sheffield]

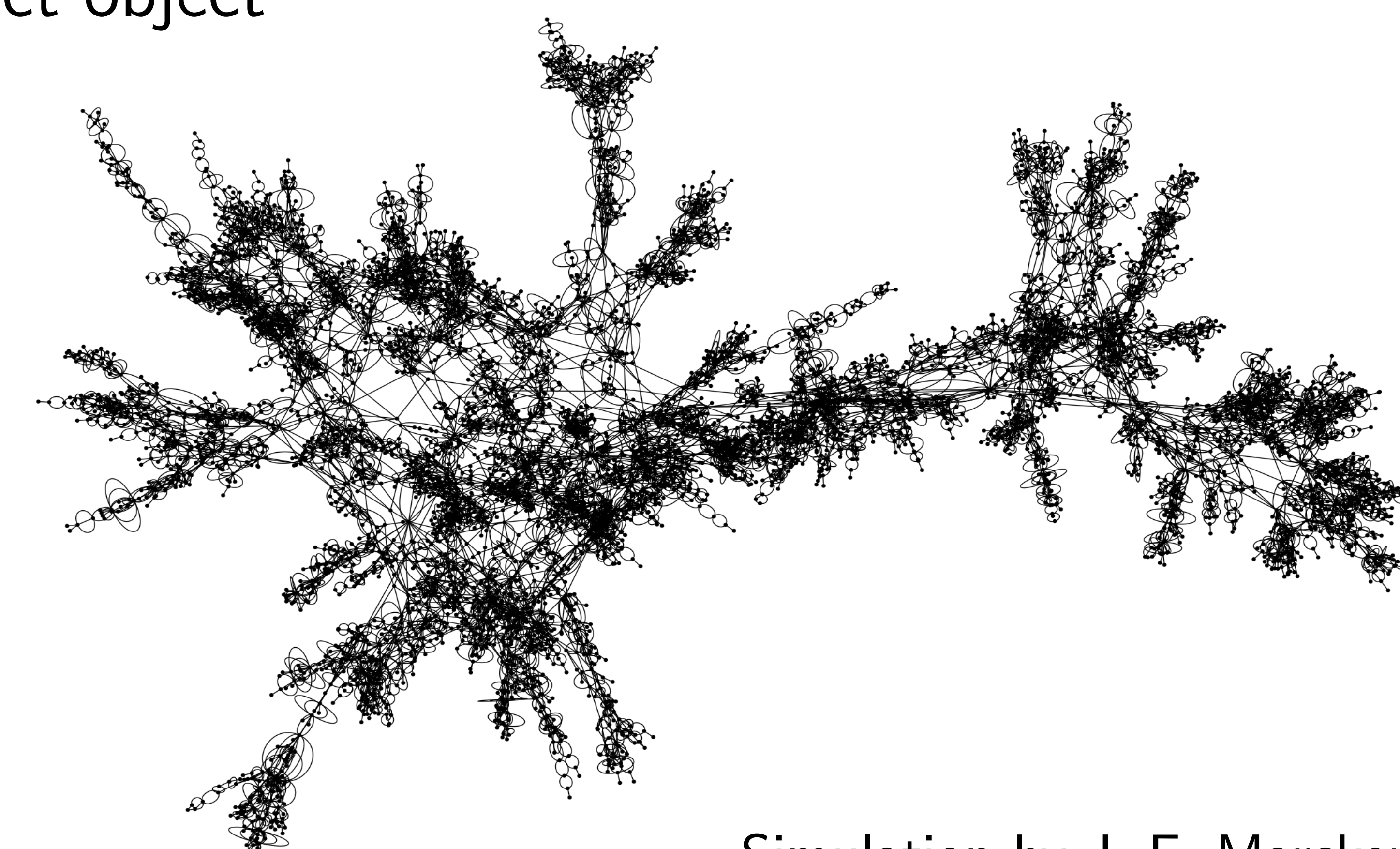
Critical exponents of some models of statistical physics (percolation, ...) are much easier to compute on random lattices (i.e. random maps) than on a fixed Euclidean lattice.

Scaling limit

- **Global point of view: convergence of the rescaled maps**

When the size of the map goes to infinity, so does the typical distance between two vertices.

Idea: "scale" the map = length of edges decreases with the size of the map.
Goal: obtain a compact object



Simulation by J.-F. Marckert

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well understood:

• Schaeffer's bijection : quadrangulations $\leftrightarrow$ labeled trees.
Label of one vertex in the tree = distance between this vertex and the root in the map.

• distance between two random points $\sim n^{1/4}$ + law of the distance [Chassaing-Schaeffer '04]

• cvgence of normalized quadrangulations + limiting object: Brownian map. [Marckert-Mokkadem '06], [Le Gall '07], [Miermont '08], [Miermont 13], [Le Gall 13]

Convergence of uniform rescaled quadrangulations

Theorem : [Miermont '13], [Le Gall '13]
(Q_n) = sequence of random quadrangulations, then:

$$\left(Q_n, \left(\frac{9}{8n}\right)^{1/4} d_{Q_n}\right) \xrightarrow{(d)} (M, D^\star),$$

for the distance of Gromov-Hausdorff on the isometry classes of compact metric spaces.

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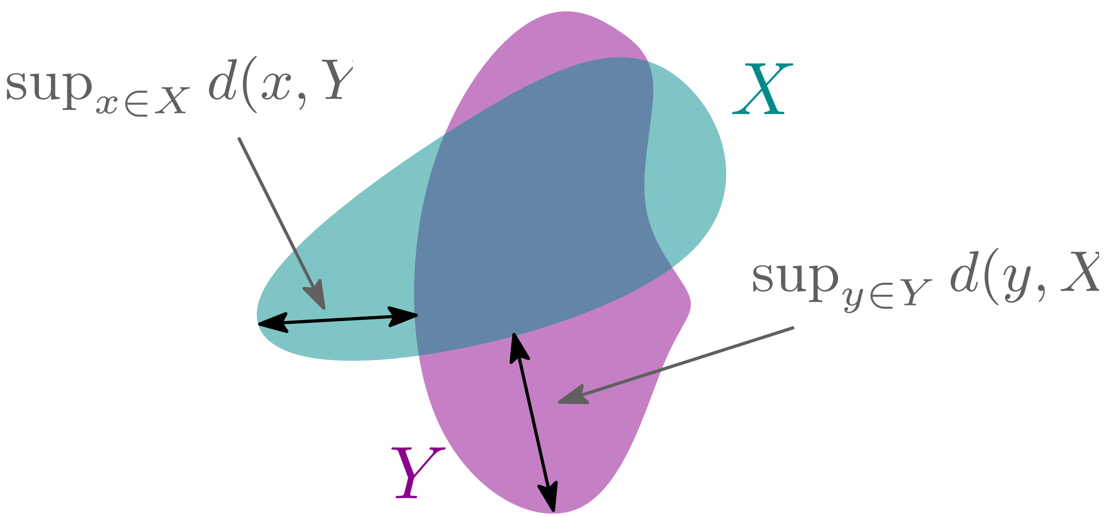
- distance between compact spaces.

Gromov-Hausdorff distance

Hausdorff distance between X and Y two compact sets of (E, d) :

$$d_H(X,Y) = \max\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\}$$



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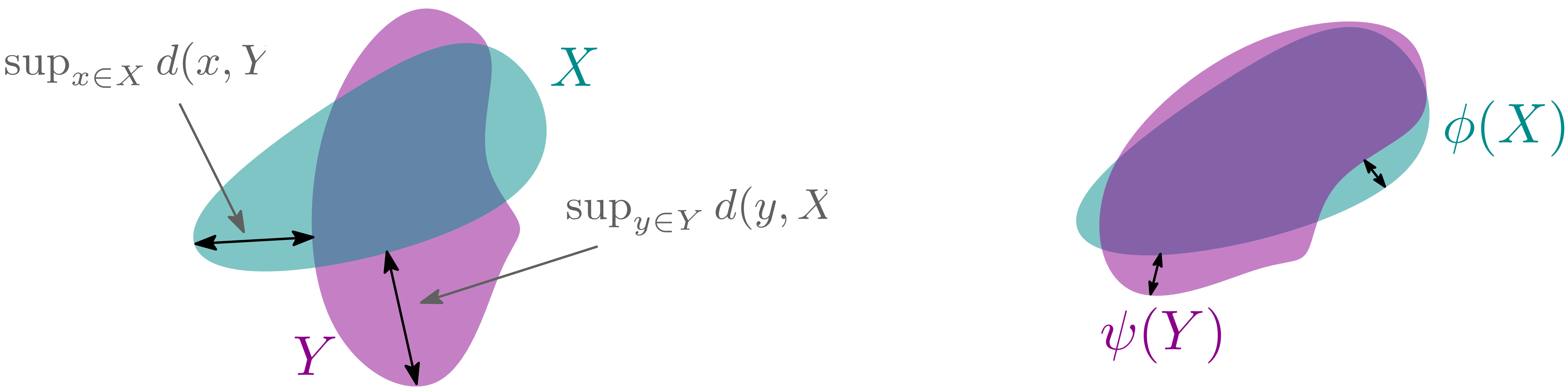

Gromov-Hausdorff distance btw two compact metric spaces E and F:

$$\mathbf{d_{GH}(E,F)} = \inf \mathbf{d_H}(\phi(\mathbf{E}), \psi(\mathbf{F}))$$

Infimum taken on :

- all the metric spaces M
- all the isometric embeddings $\phi, \psi : E, F \to M$.

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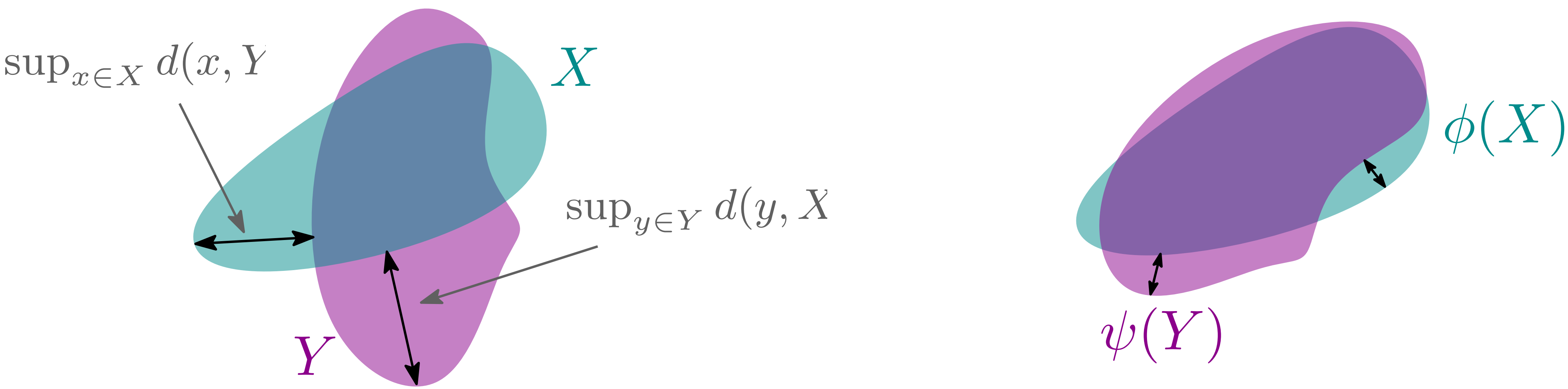

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{isometry classes of compact metric spaces with GH distance}
$=$ complete and separable ($=$ "Polish") space.

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- The Brownian Map

Universality

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Idea : The Brownian map is a **universal** limiting object.
All "reasonable models" of maps (properly rescaled) are expected to converge towards it.

Problem : The results of Miermont and Le Gall rely on nice bijections between maps and labeled trees [Schaeffer '98], [Bouttier-Di Francesco-Guitter '04].

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The result

Theorem : [Addario-Berry, A.]

$(M_n) =$ sequence of random **simple** triangulations, then:

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Exactly the same kind of result as Le Gall and Miermont's.

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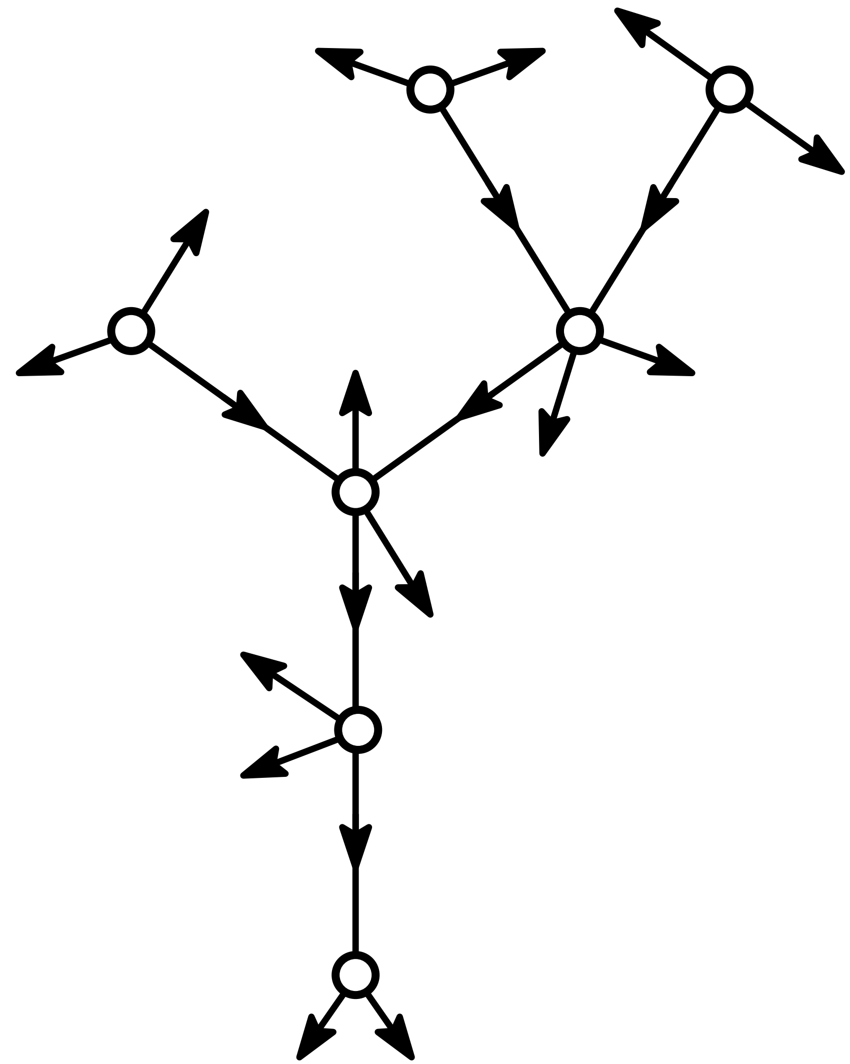
Idea of proof :

- encode the simple triangulations by some trees,
- **study the limits of trees**,
- interpret the **distance in the maps by some function of the tree**.

From blossoming trees to simple triangulations

plane tree:
plane map that is a tree



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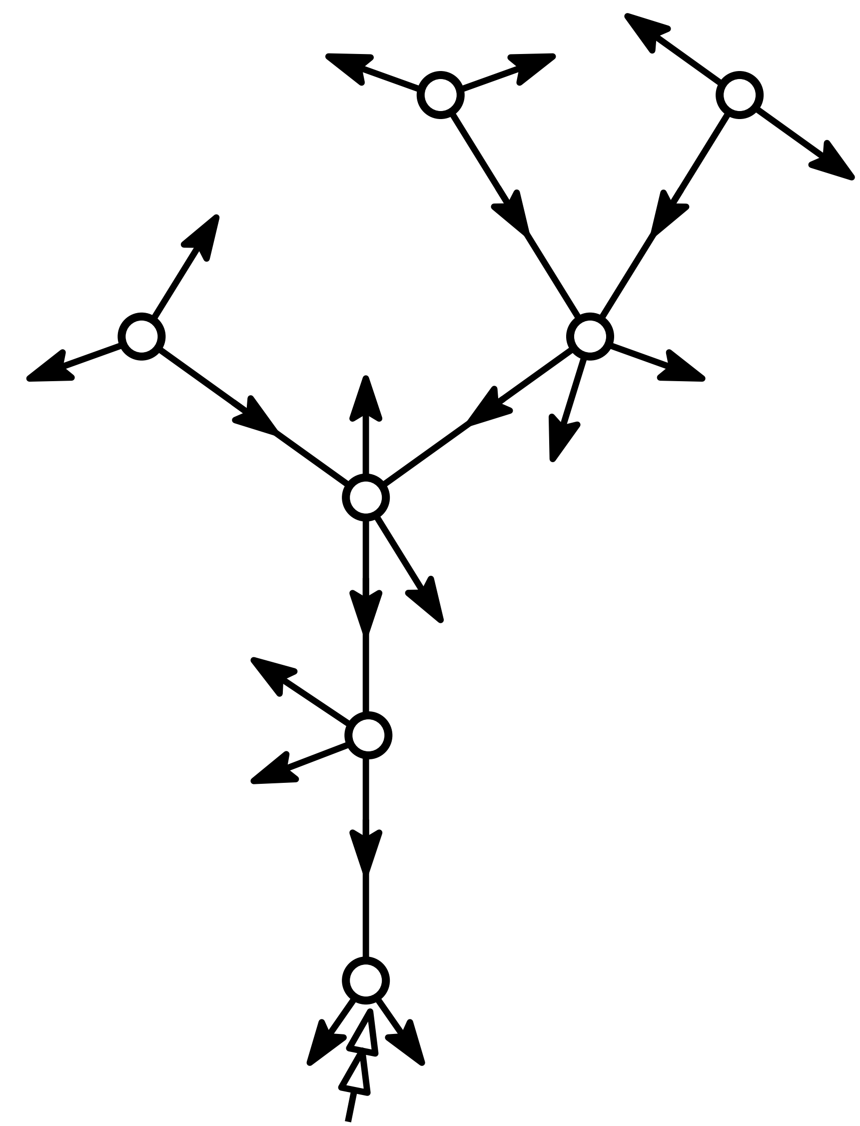

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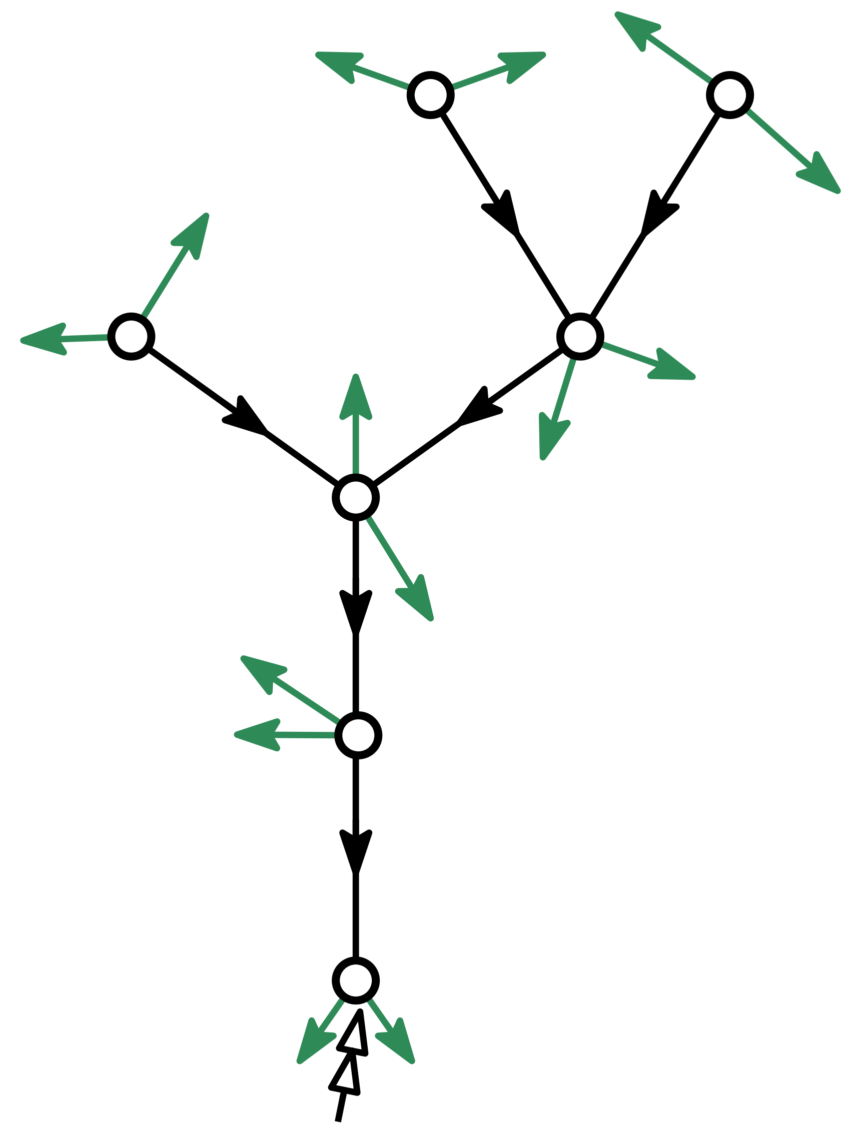
2-blossoming tree:
planted plane tree such that each vertex carries two leaves



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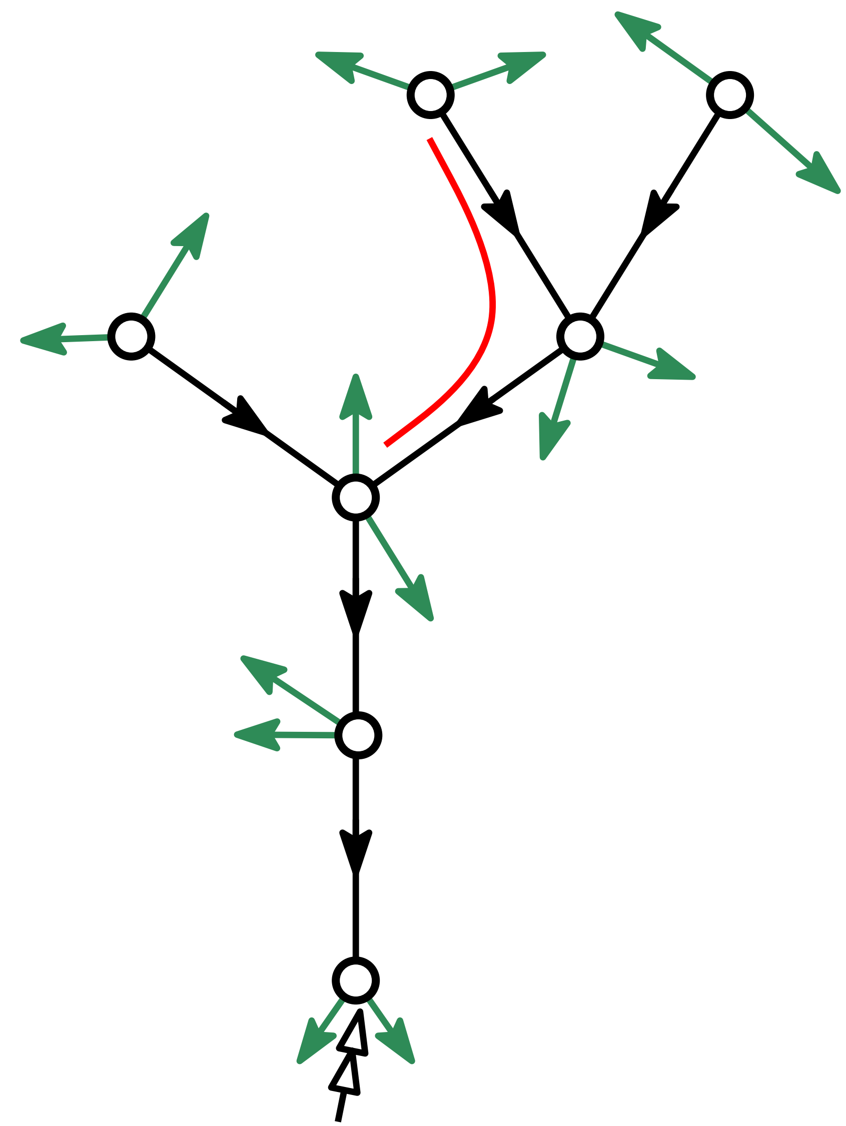
Given a planted 2-blossoming tree:

- If a leaf is followed by two internal edges,



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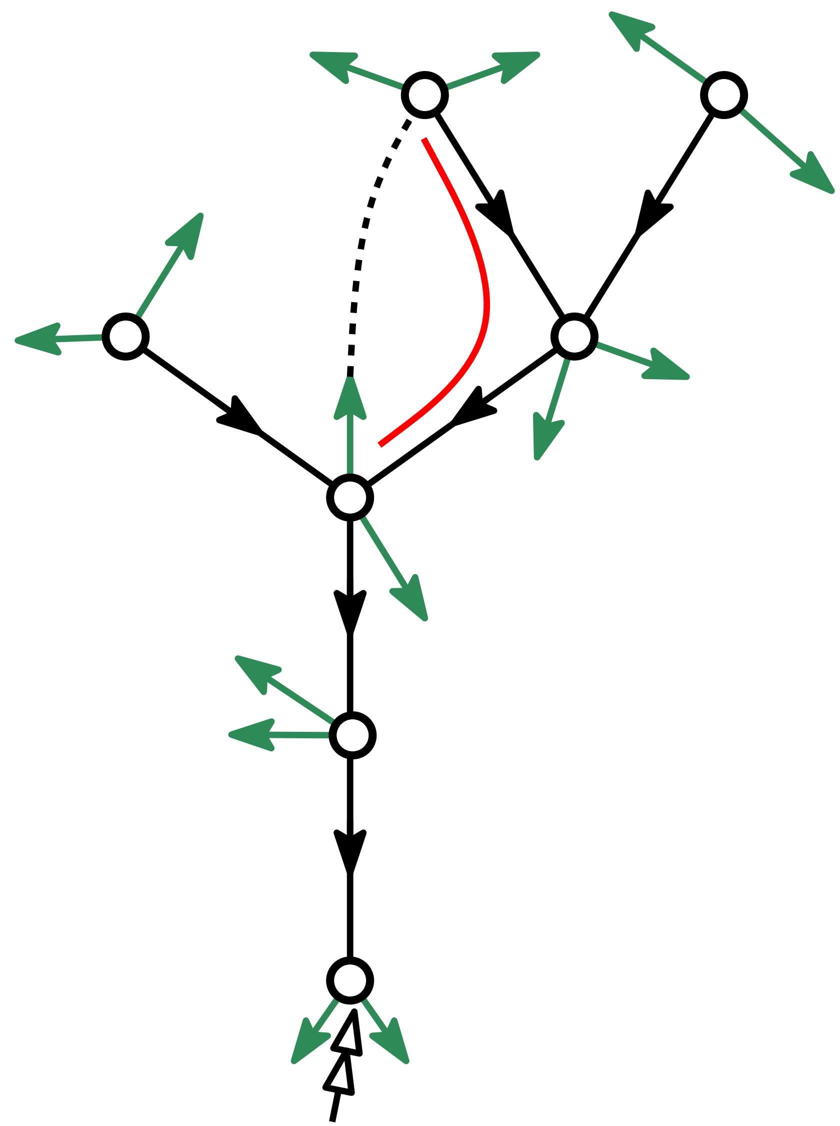

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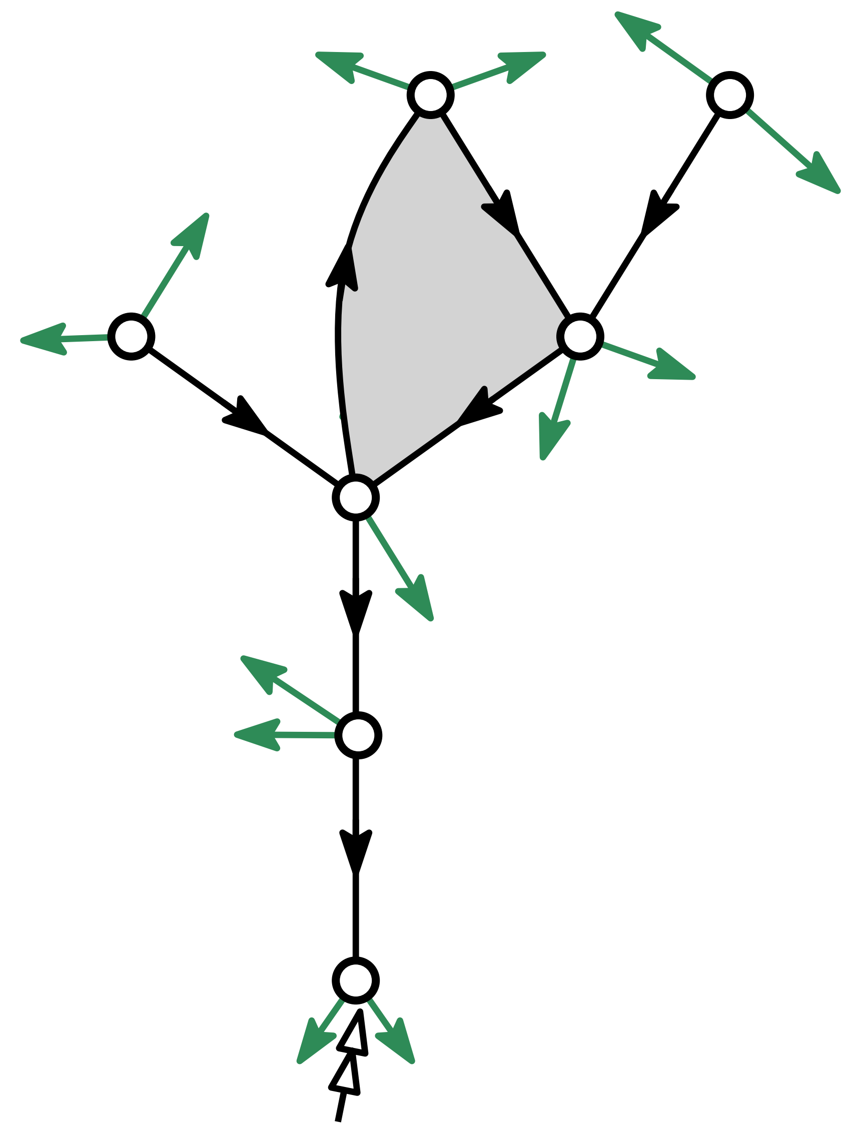

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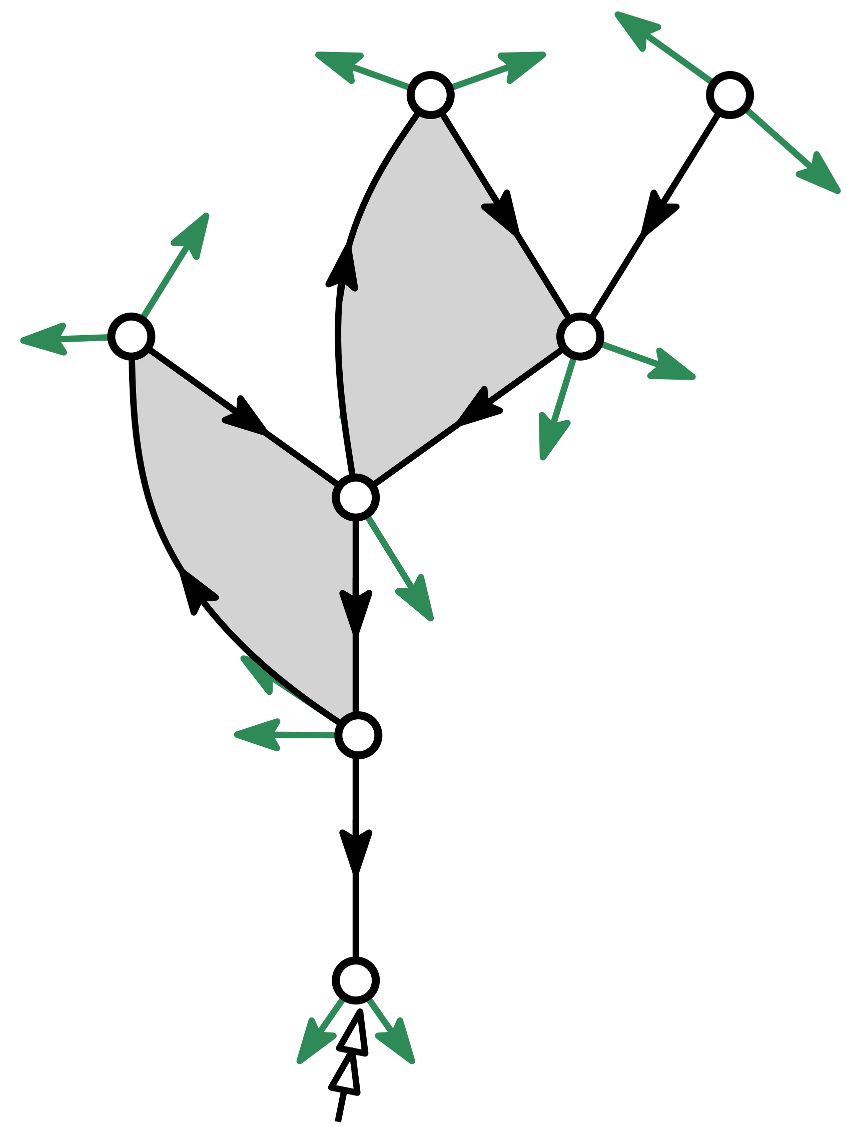

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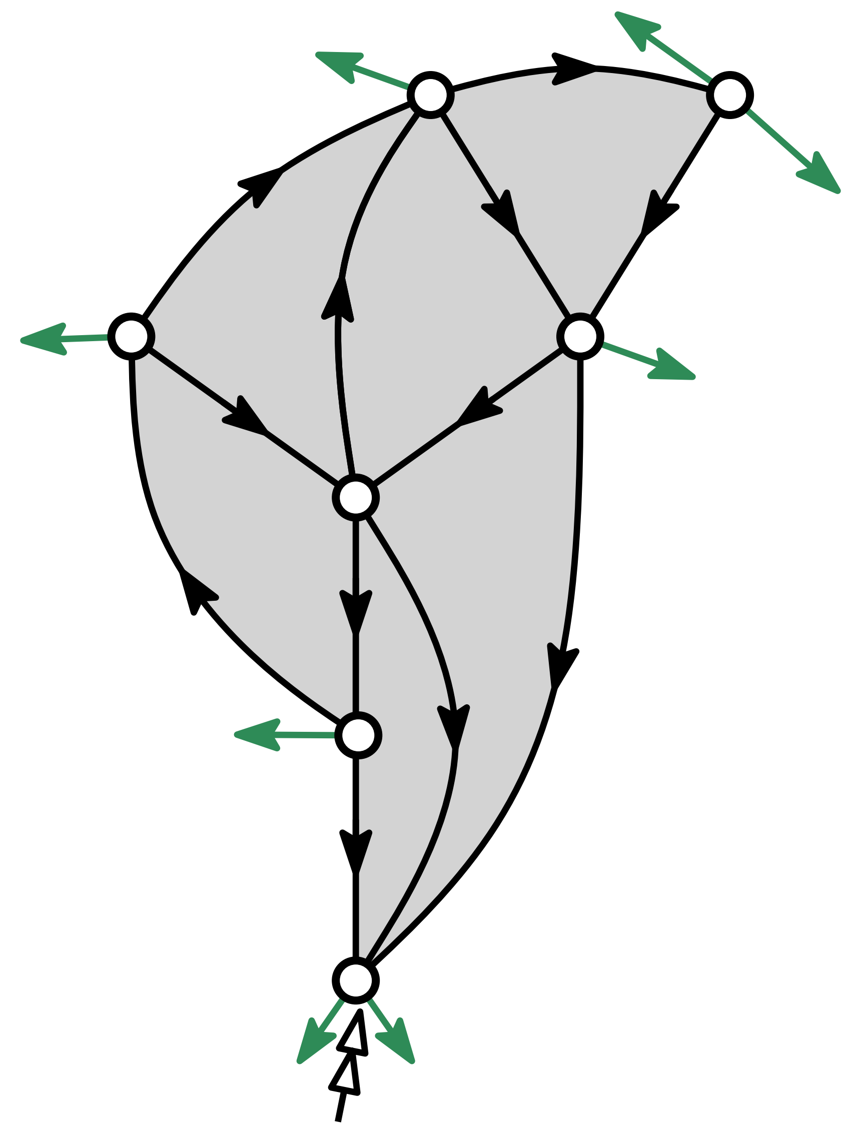

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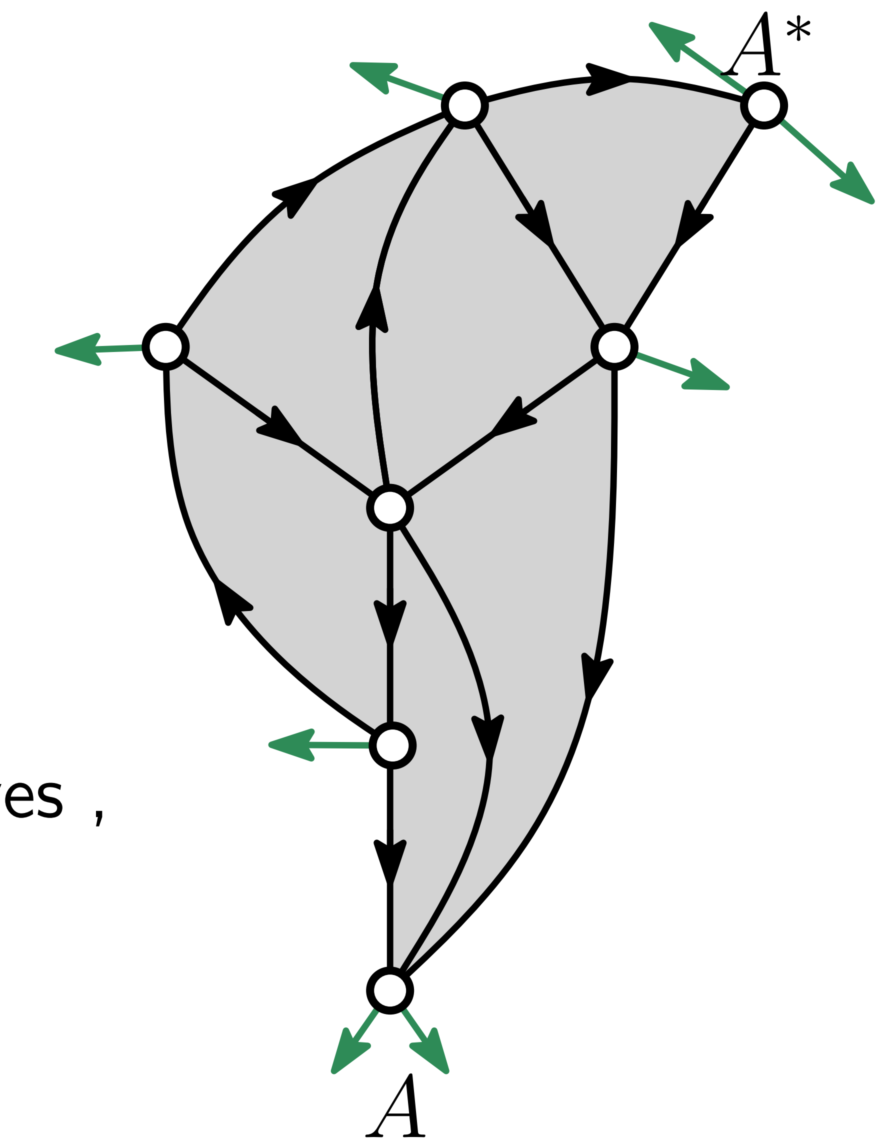

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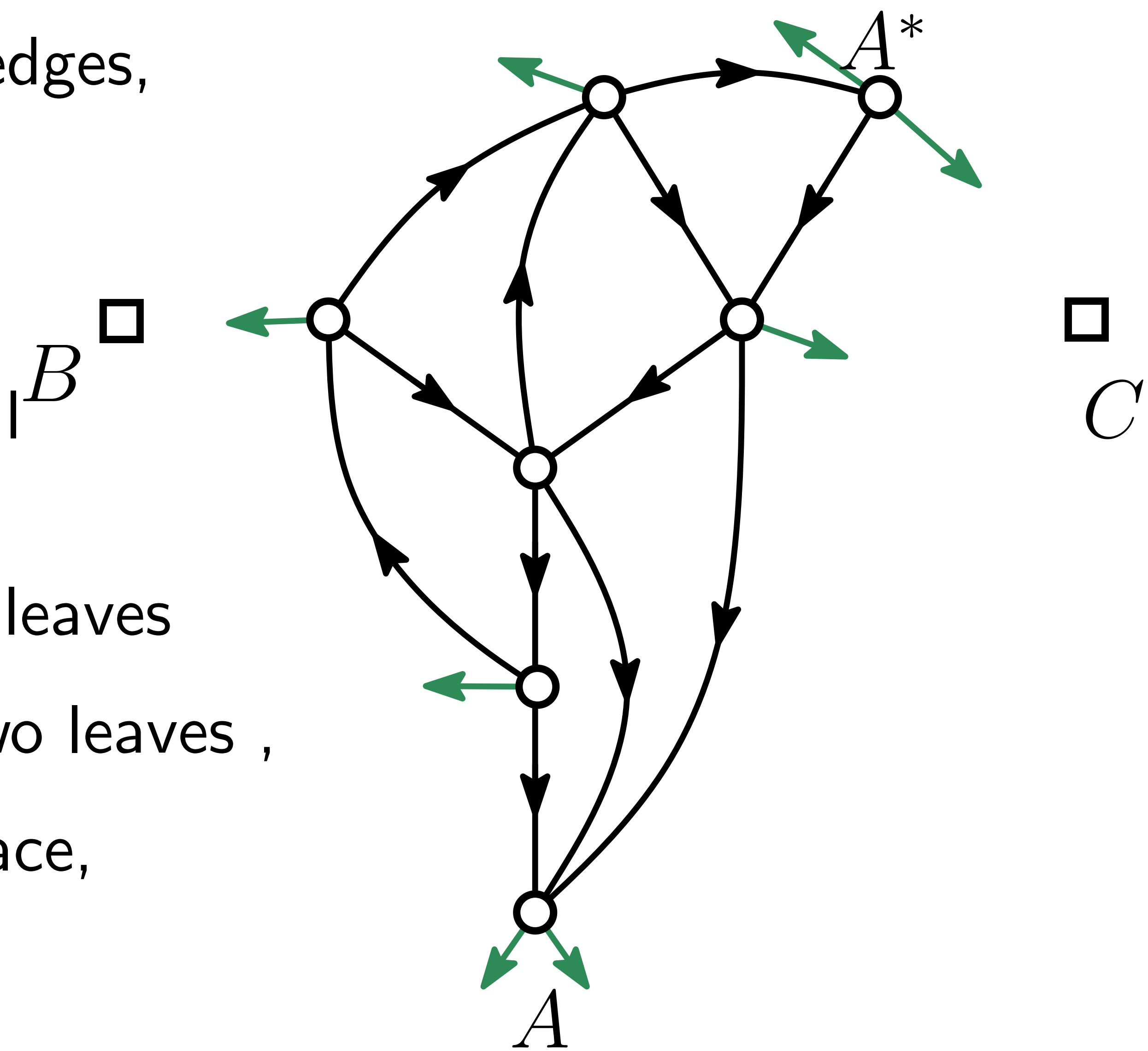
- Add two new vertices in the outer face,



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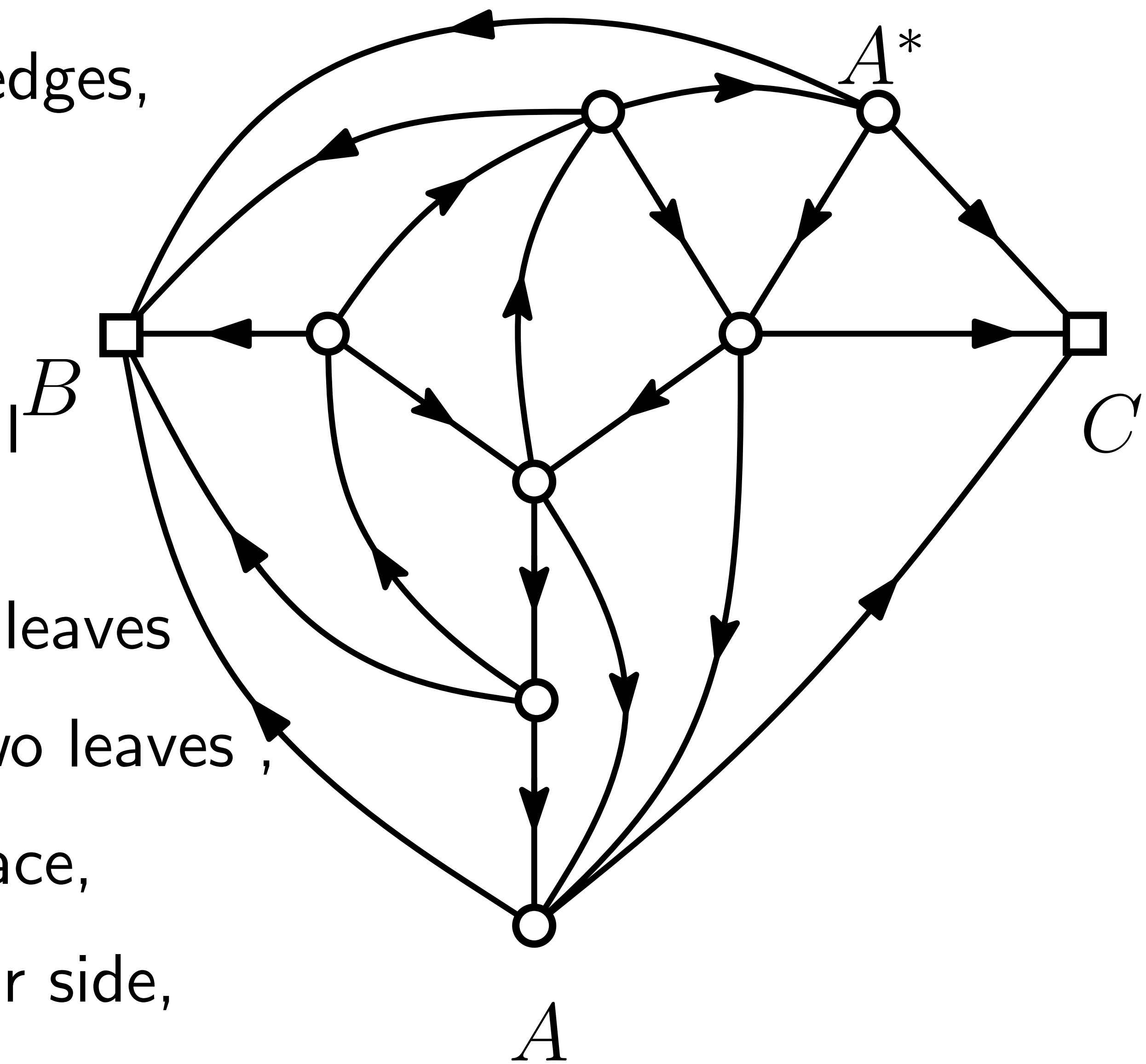
- label A and $A^\star$, the vertices with two leaves ,
- Add two new vertices in the outer face,
- Connect leaves to the vertex on their side,



From blossoming trees to simple triangulations

Given a planted 2-blossoming tree:

- If a leaf is followed by two internal edges,
- close it to make a triangle.
- and repeat !

When finished two vertices have still two leaves and others have one.

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- label A and $A^\star$, the vertices with two leaves ,
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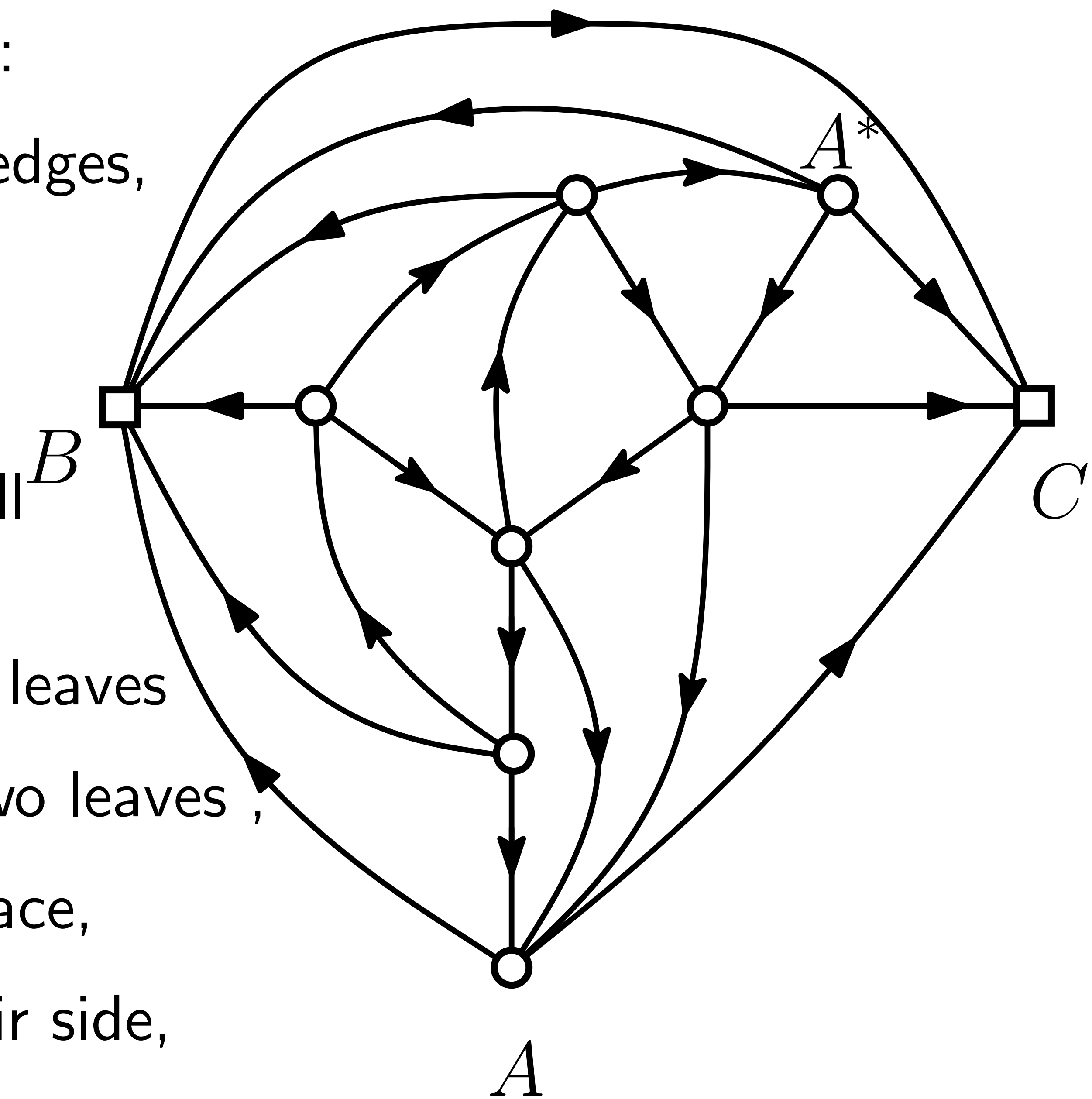
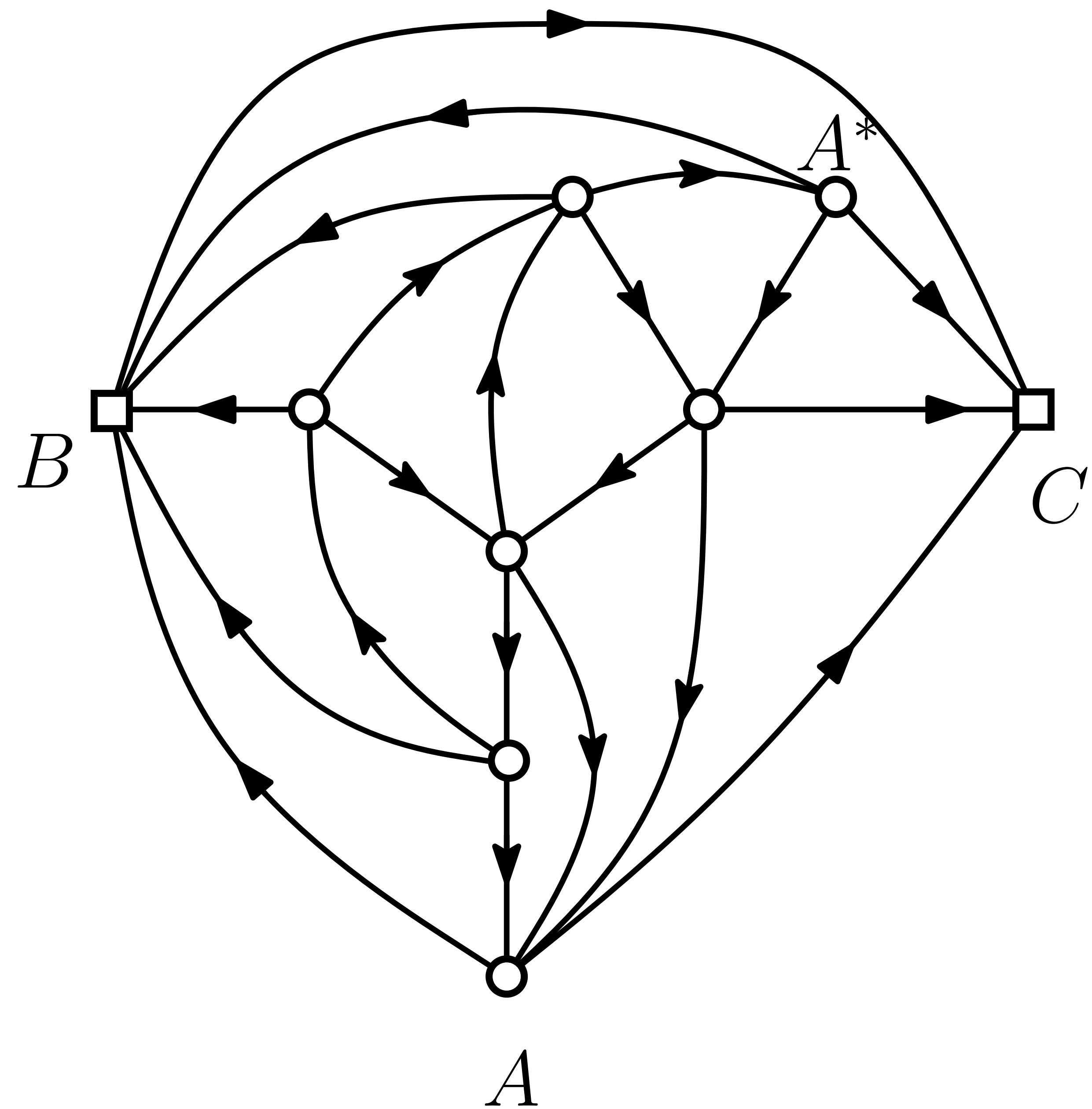
- Connect B and C.



From blossoming trees to simple triangulations

Simple triangulation endowed with its unique orientation such that :

- out$(v) = 3$ for v an inner vertex
- out$(A) = 2$, out$(B) = 1$ and out$(C) = 0$
- no counterclockwise cycle



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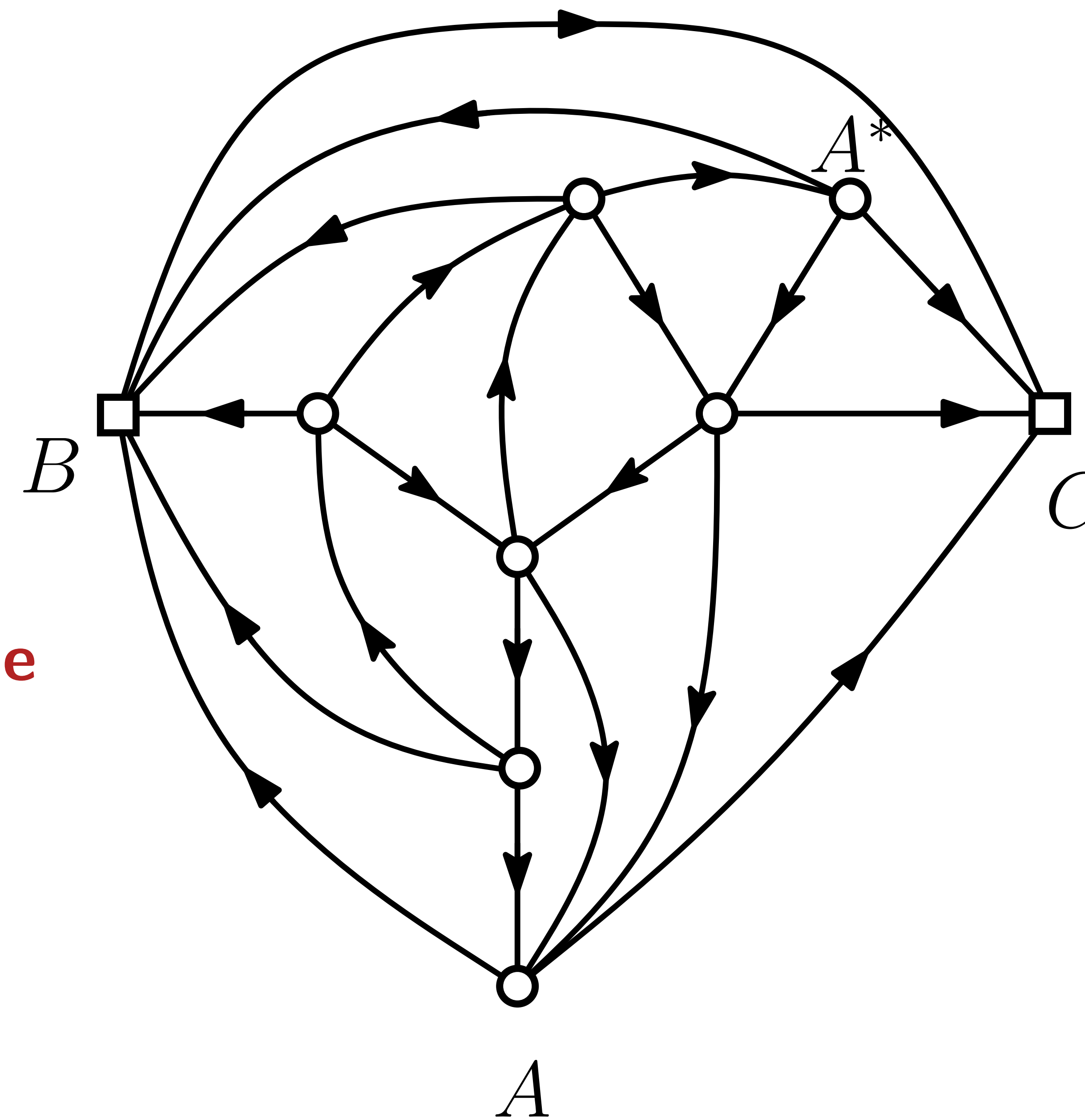
The orientations characterize simple triangulations [Schnyder]



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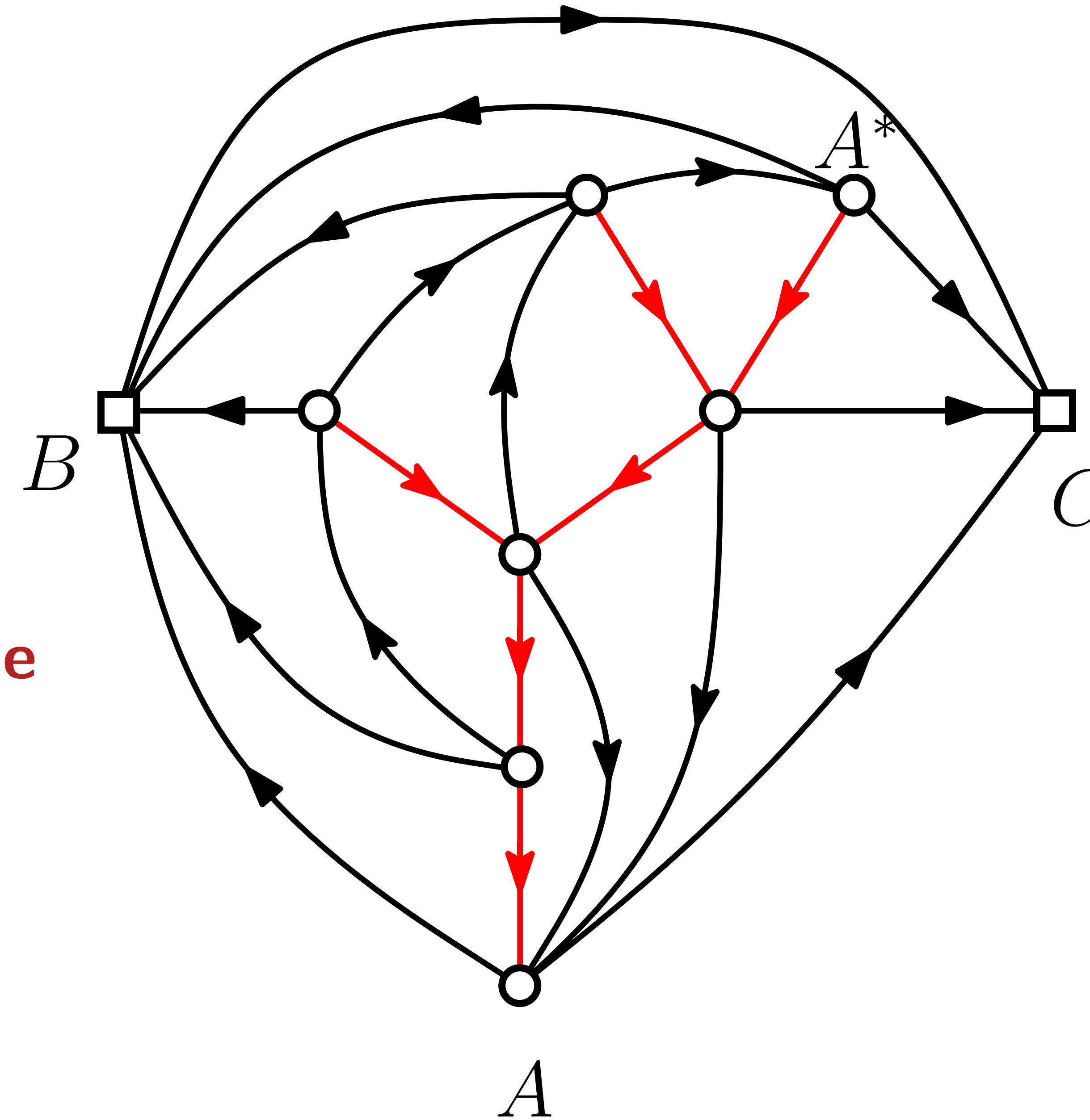
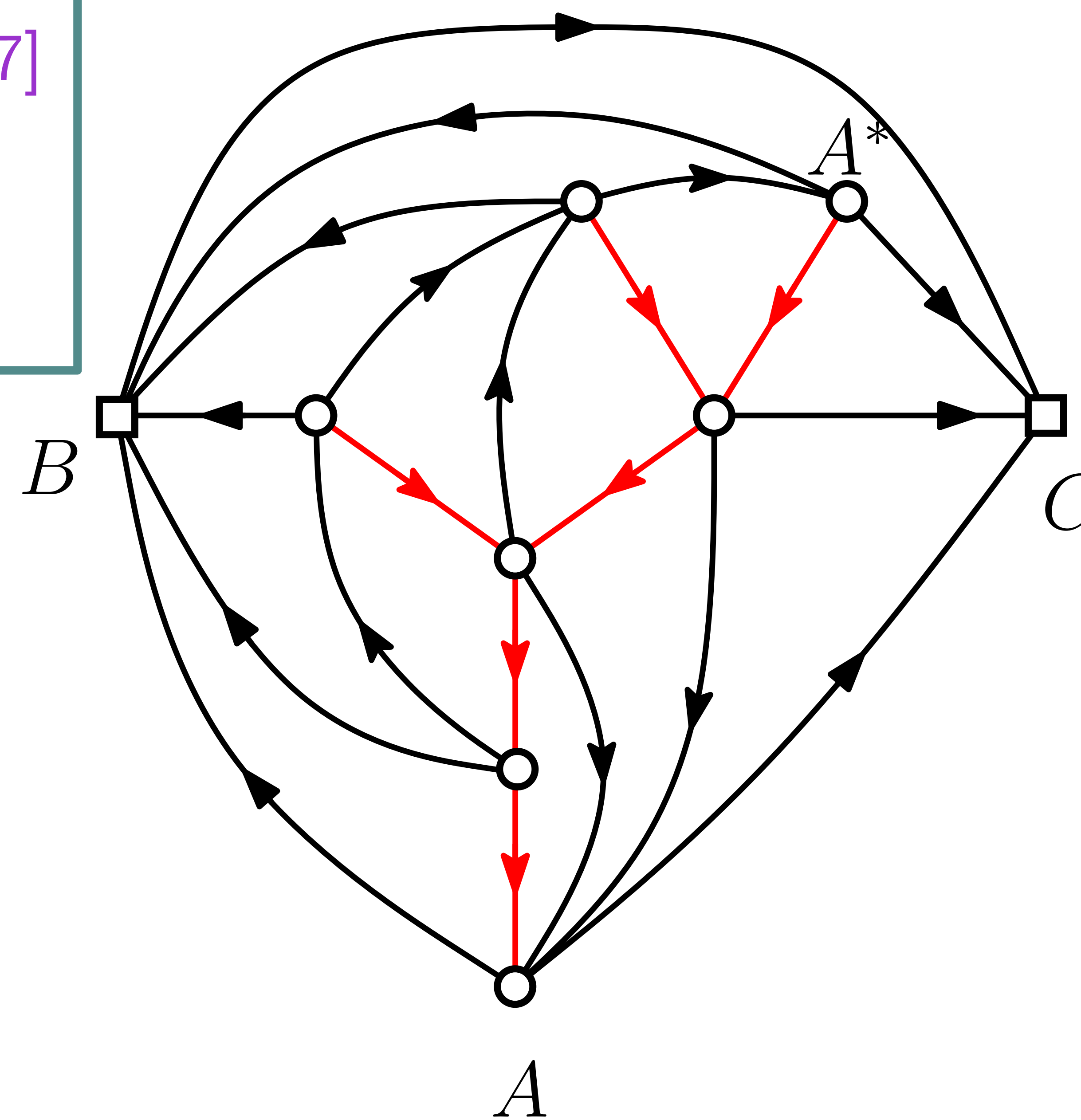
Given the orientation the blossoming tree is the leftmost spanning tree of the map (after removing B and C).



From blossoming trees to simple triangulations

Proposition: [Poulalhon, Schaeffer '07]

The closure operation is a bijection between balanced 2-blossoming trees and simple triangulations.



Same bijection with corner labels

- Start with a planted 2-blossoming tree.
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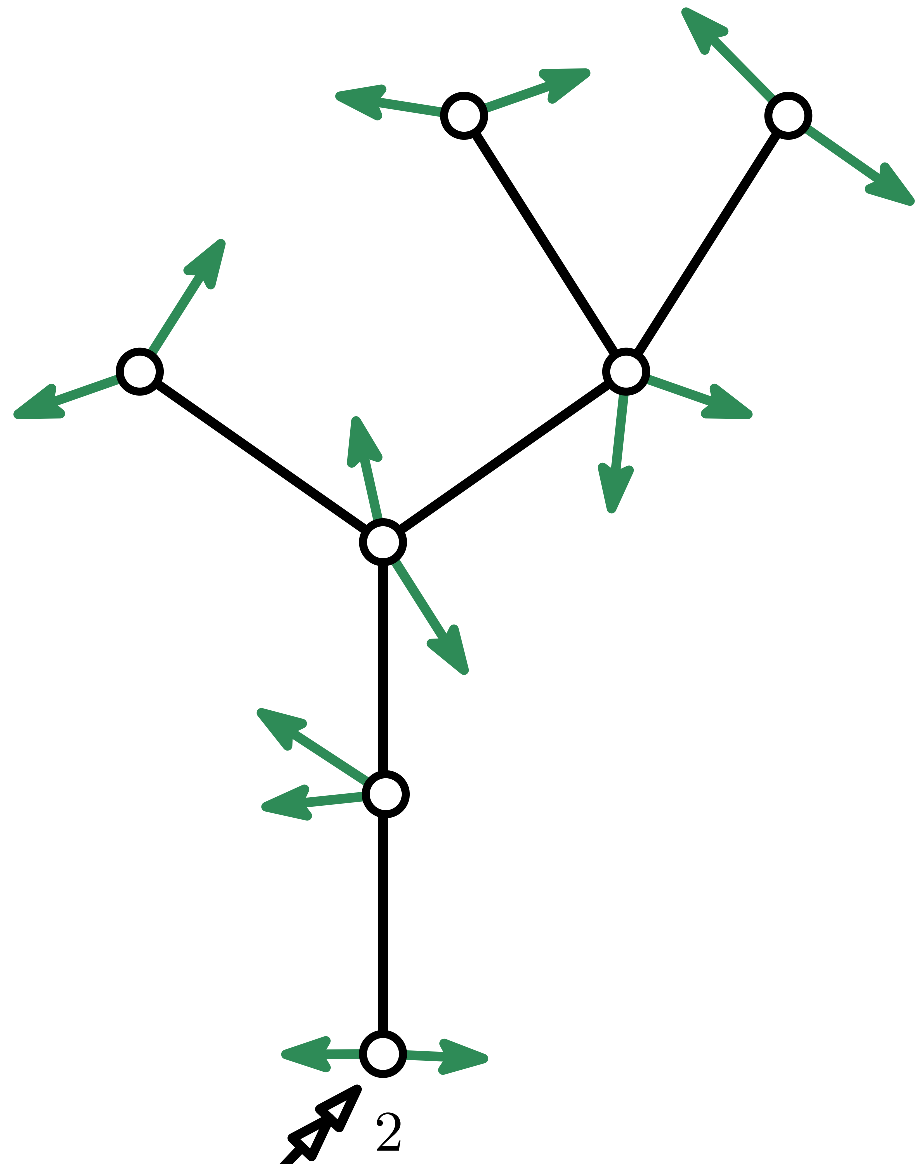

Same bijection with corner labels

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In contour order, apply the following rules:

- Non-leaf to leaf, label does not change.
- Leaf to non-leaf, label increases by 1.
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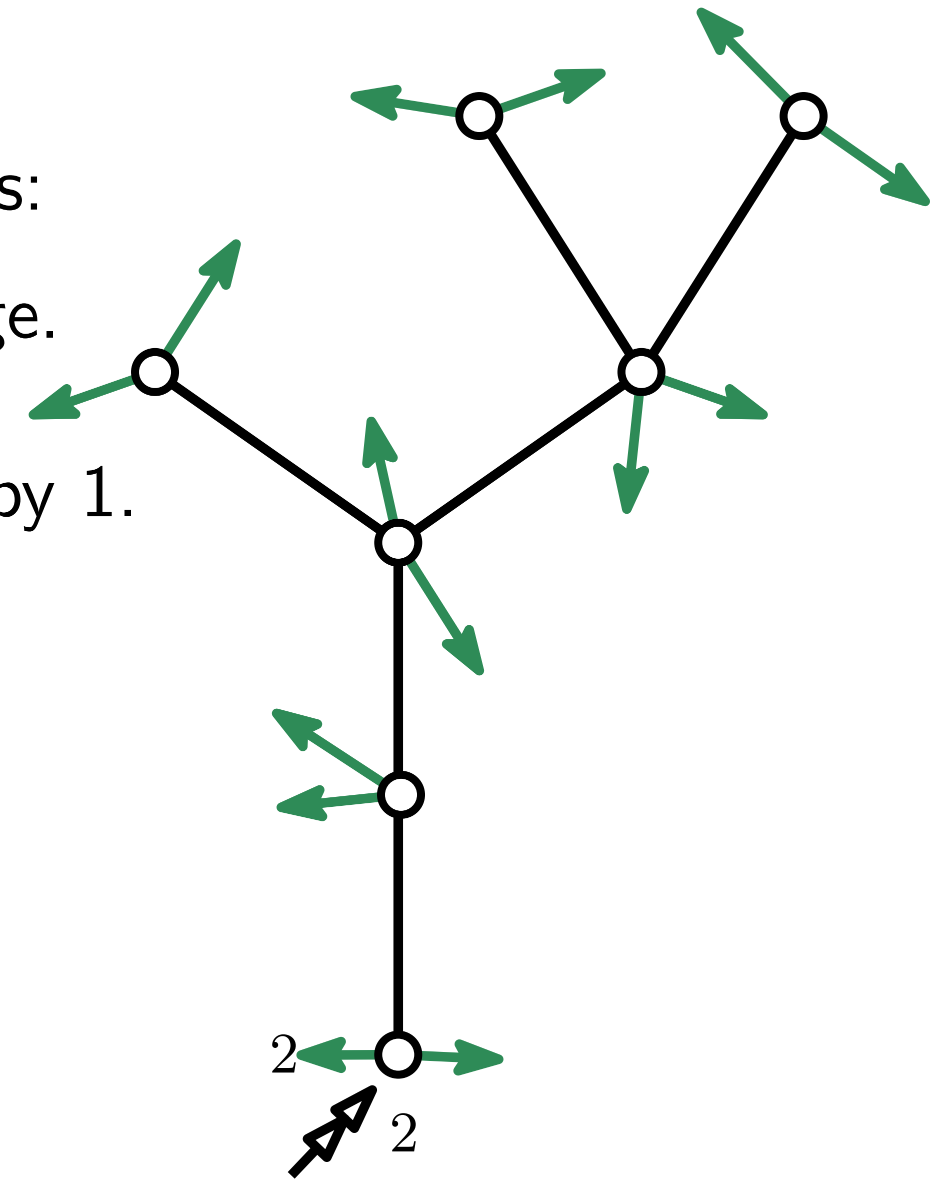

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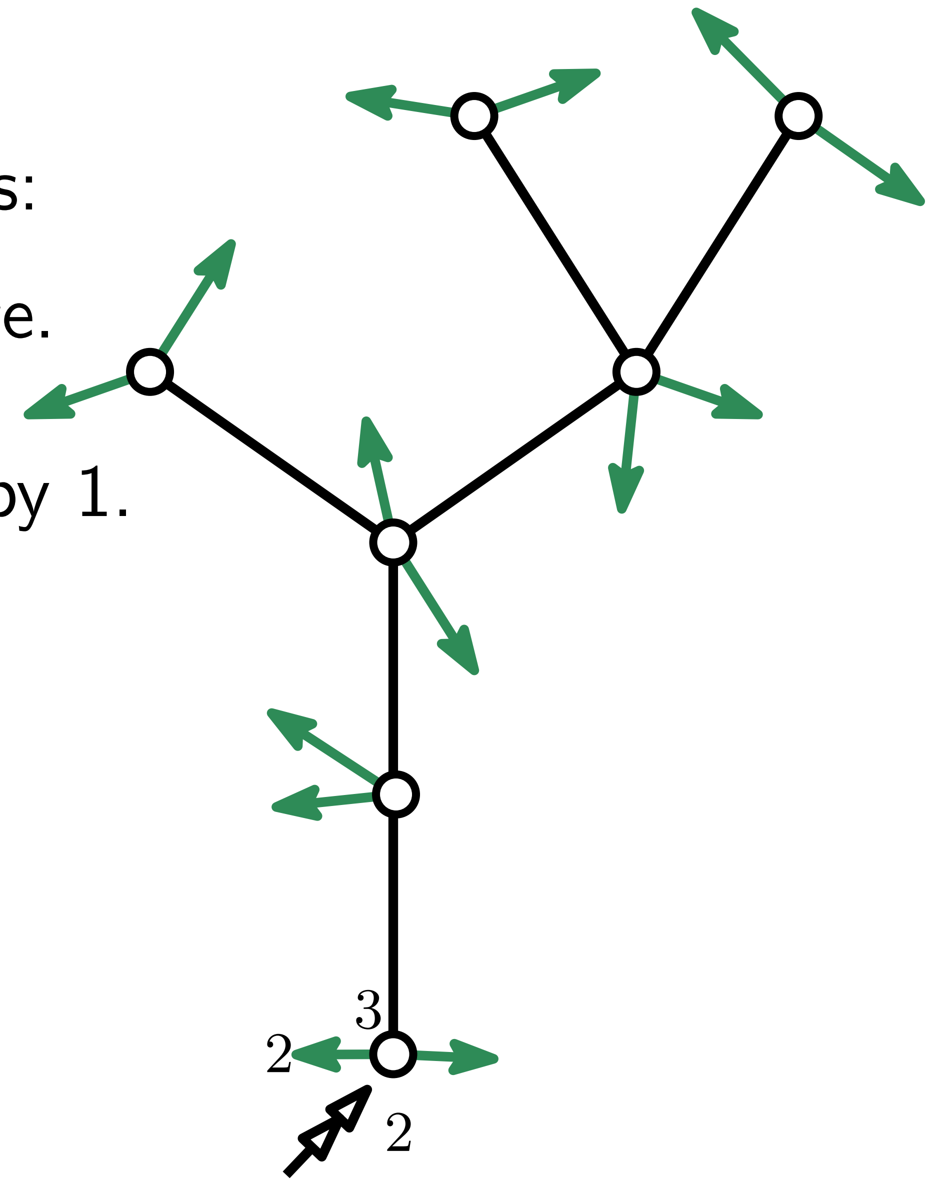

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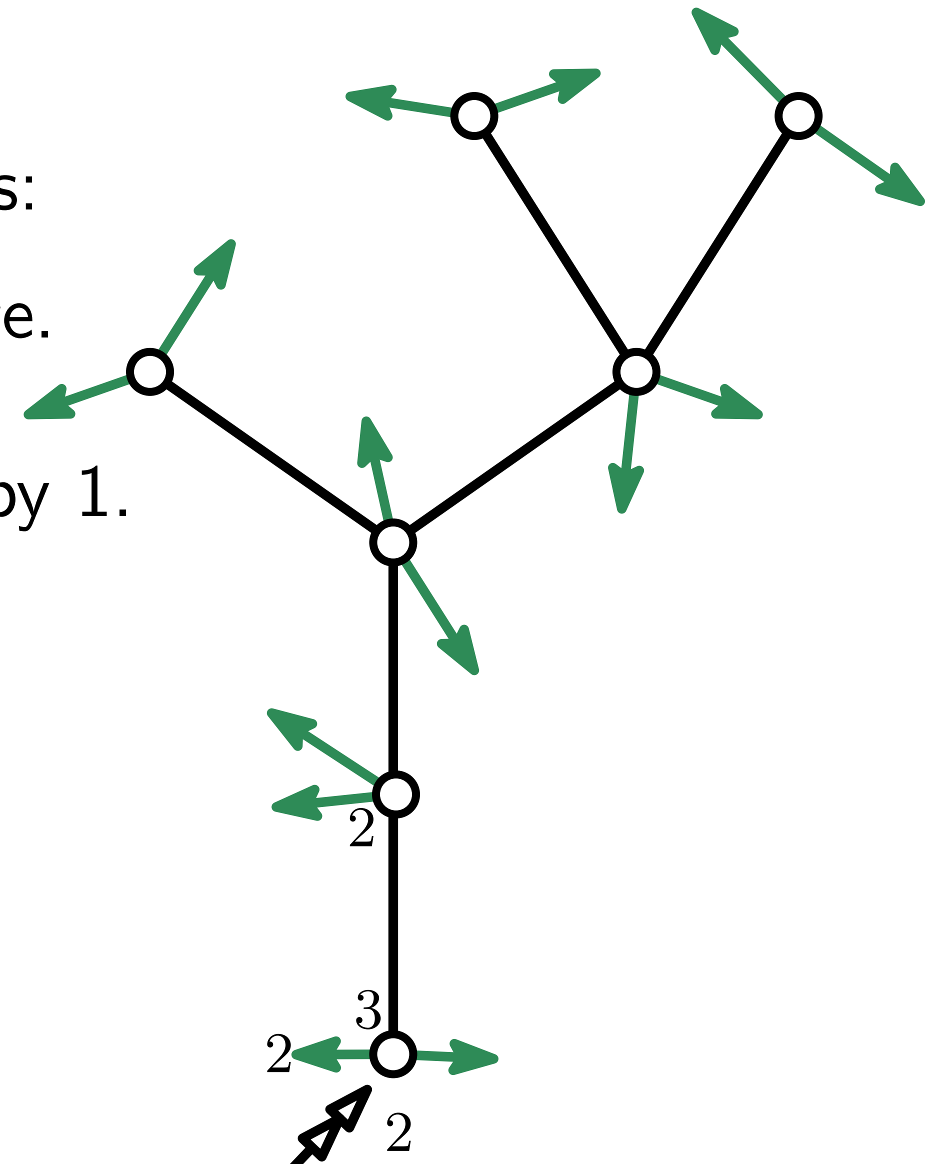

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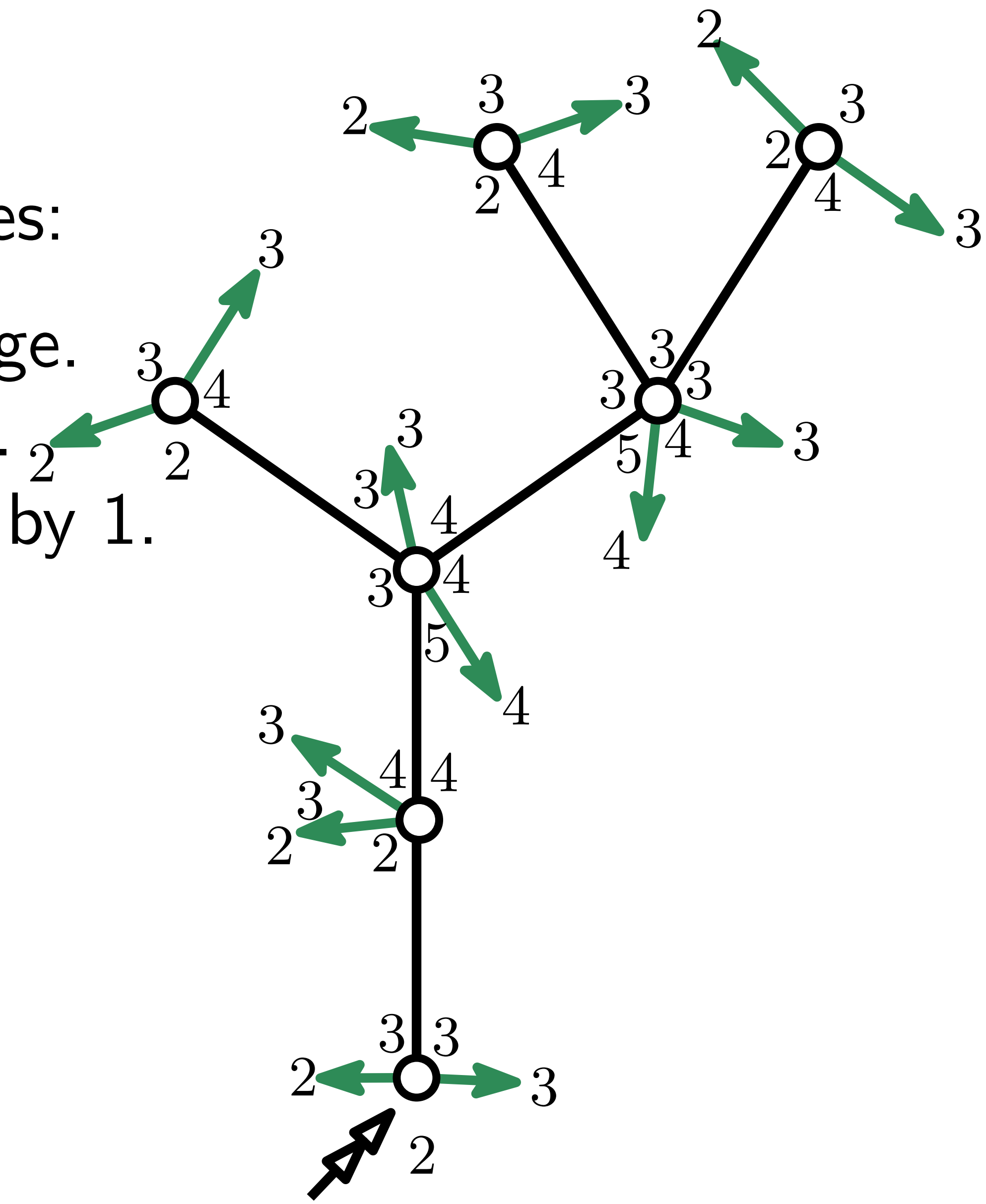

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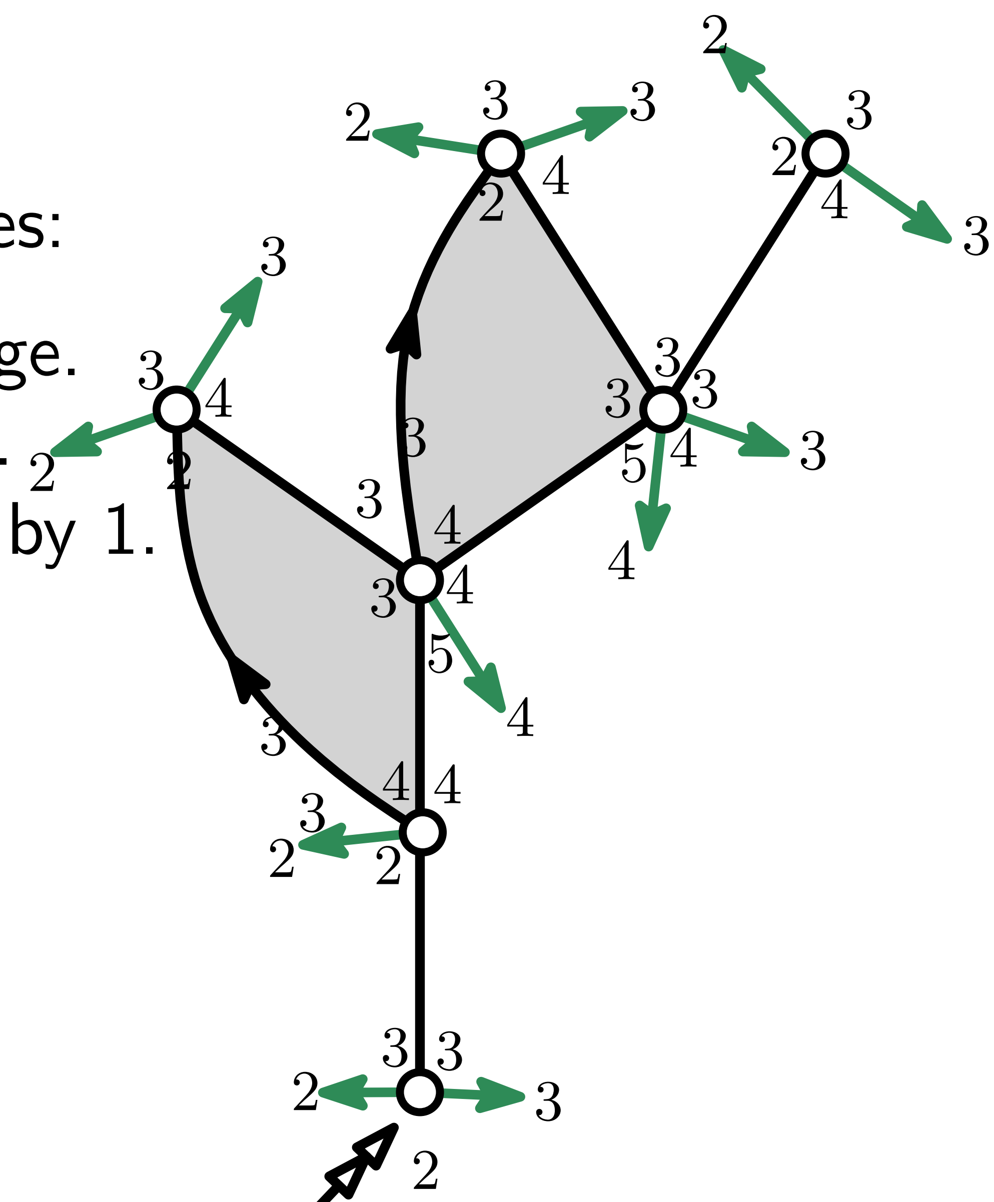

Aside: Tree is balanced $\Leftrightarrow$
all labels ≥ 2
$+$root corner incident to two stems

Closure: Merge each leaf with the first subsequent corner with a smaller label.

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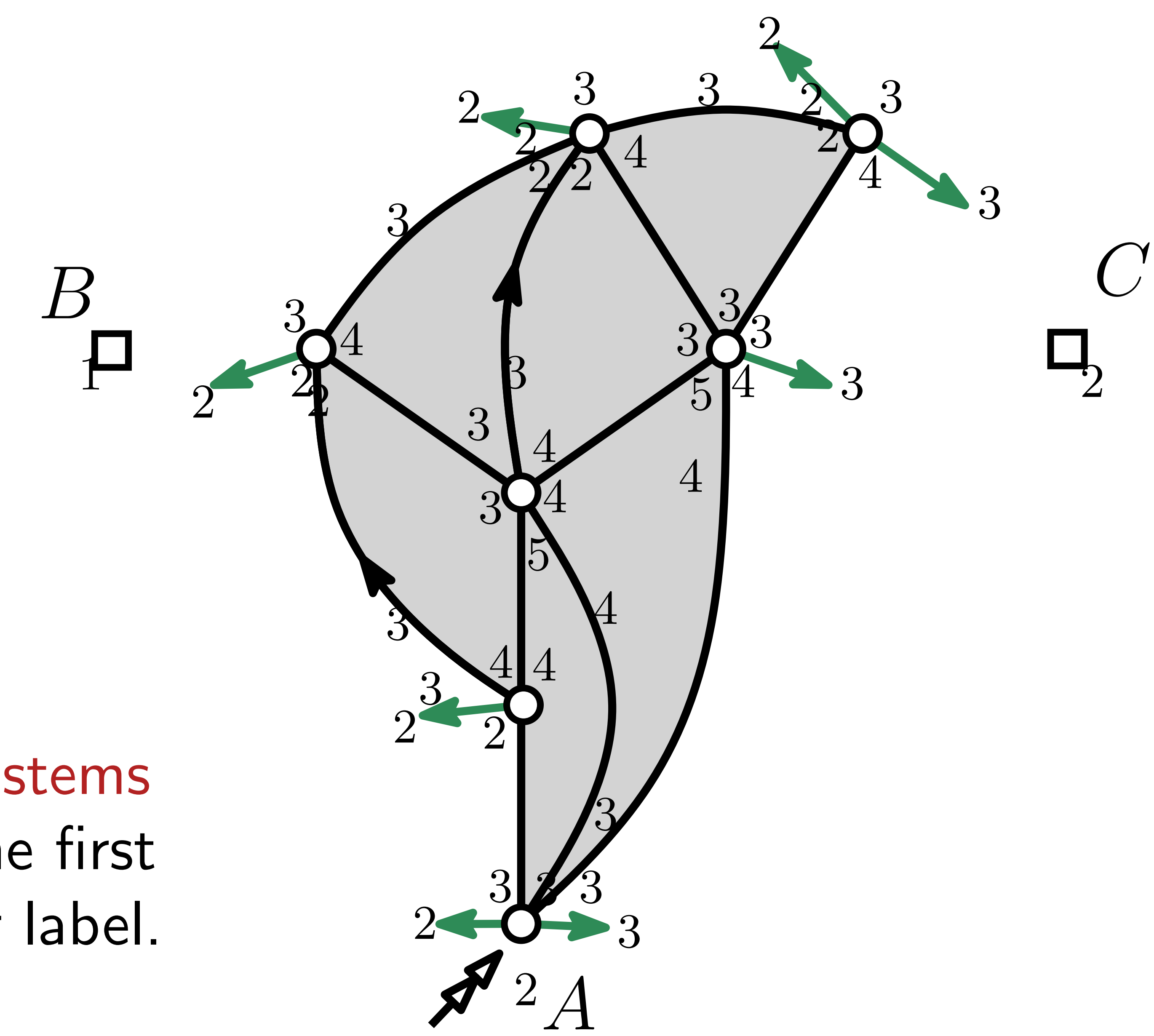

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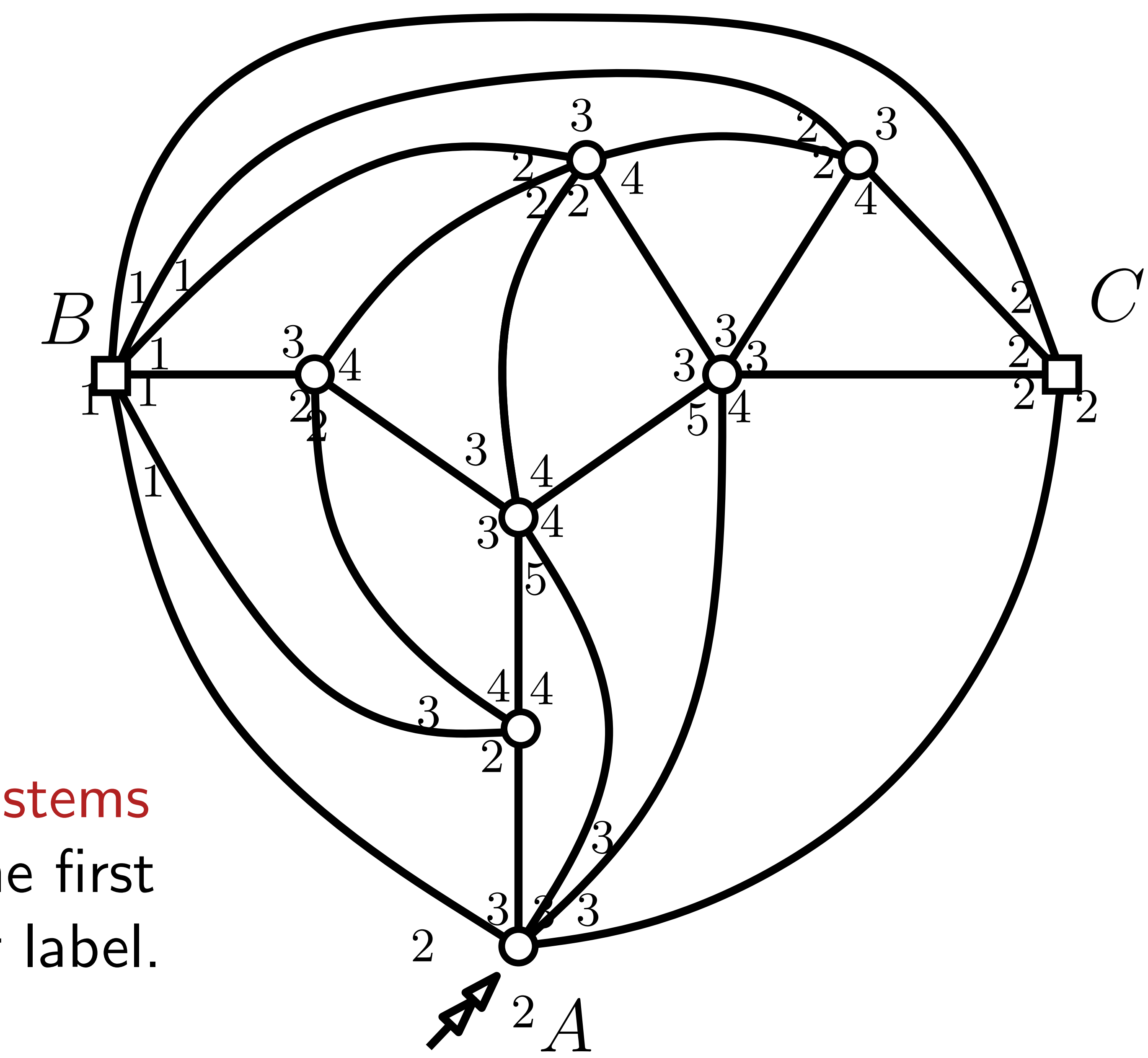

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Convergence of labeled trees

Theorem : [Addario-Berry, A.]
For a sequence of simple random triangulations (M_n), the contour and label process of the associated labeled tree satisfie:

$$\left((3n)^{-1/2}C_{\lfloor nt \rfloor}, (4n/3)^{-1/4}\tilde{Z}_{\lfloor nt \rfloor}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (e_t, Z_t)_{0\leq t\leq 1},$$

Contour and label processes of a labeled tree

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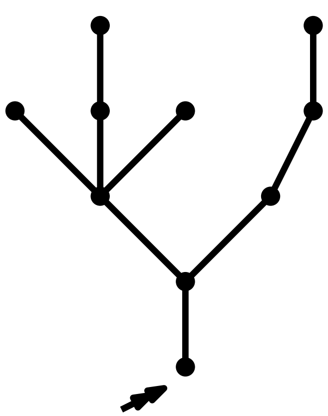

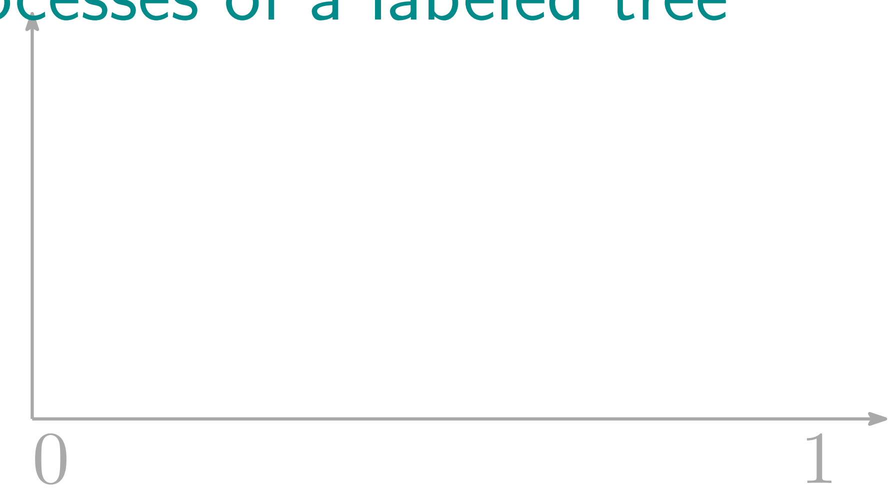

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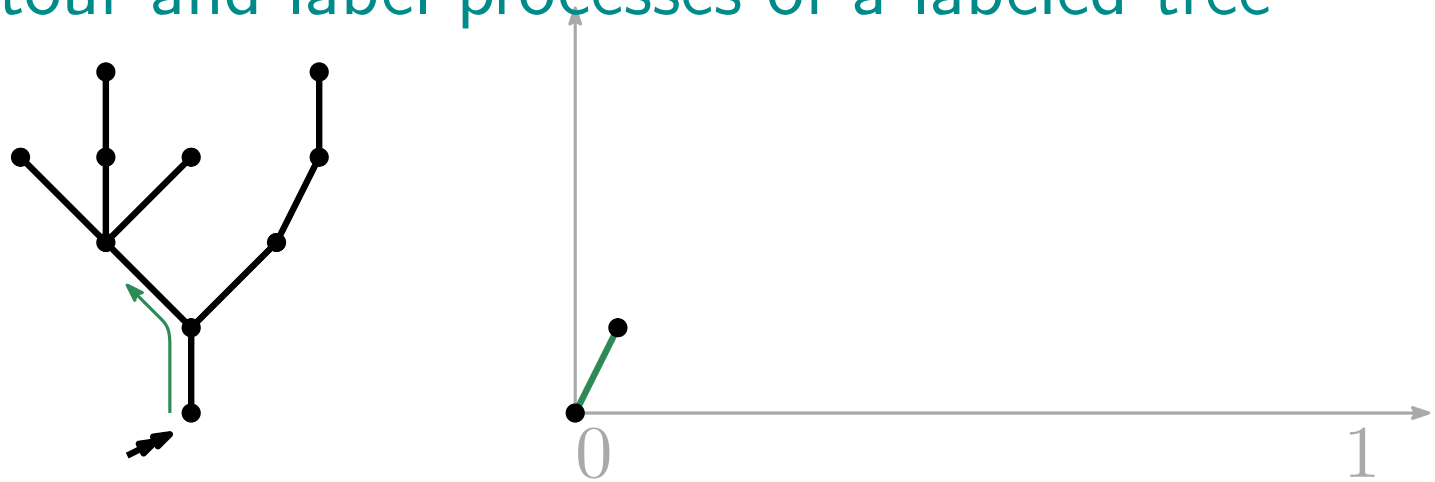

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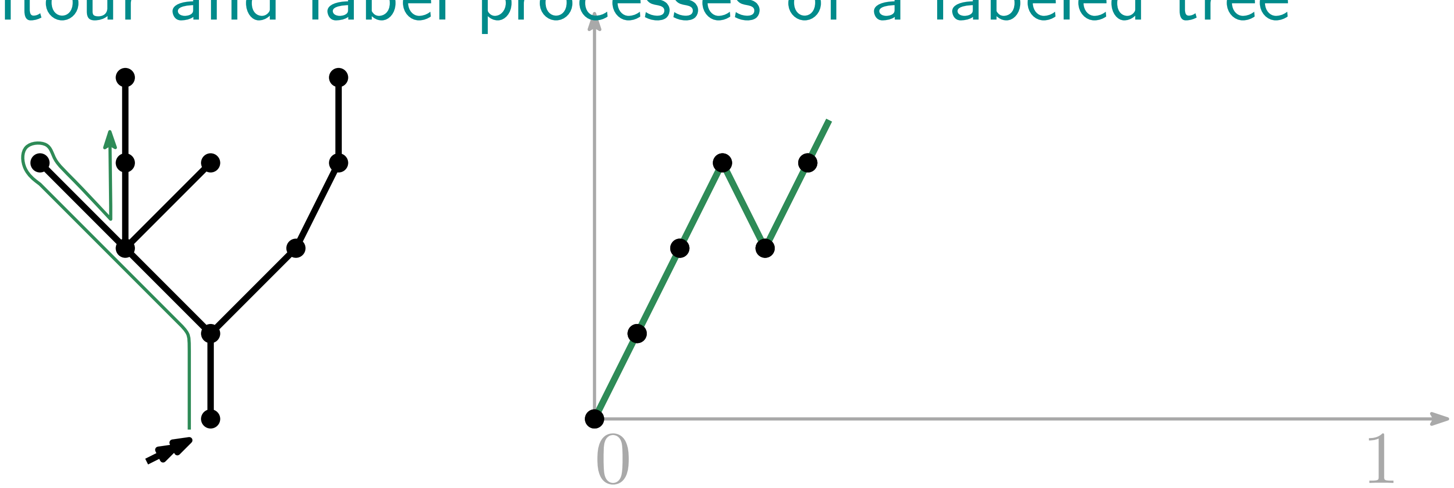

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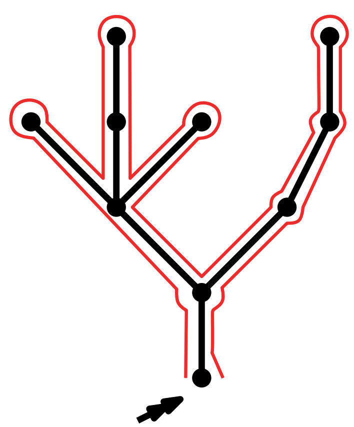

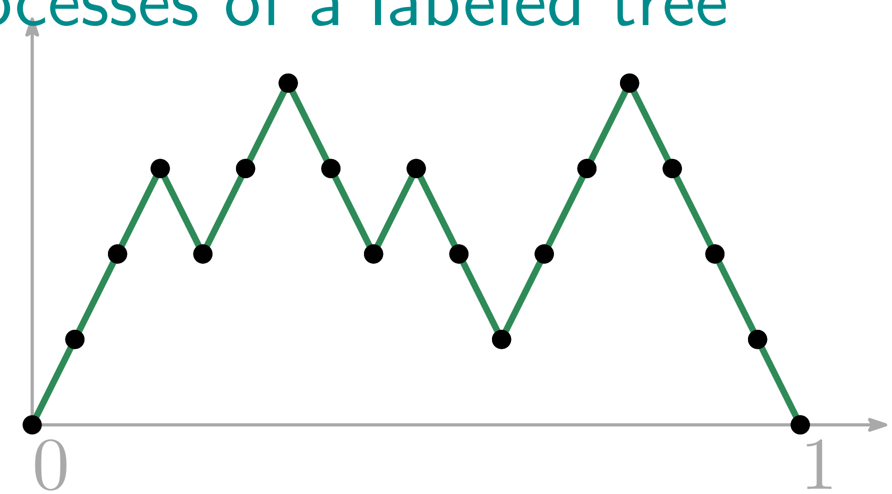

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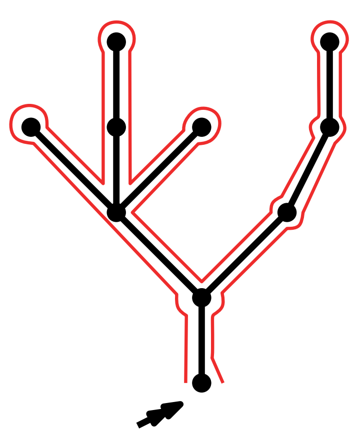

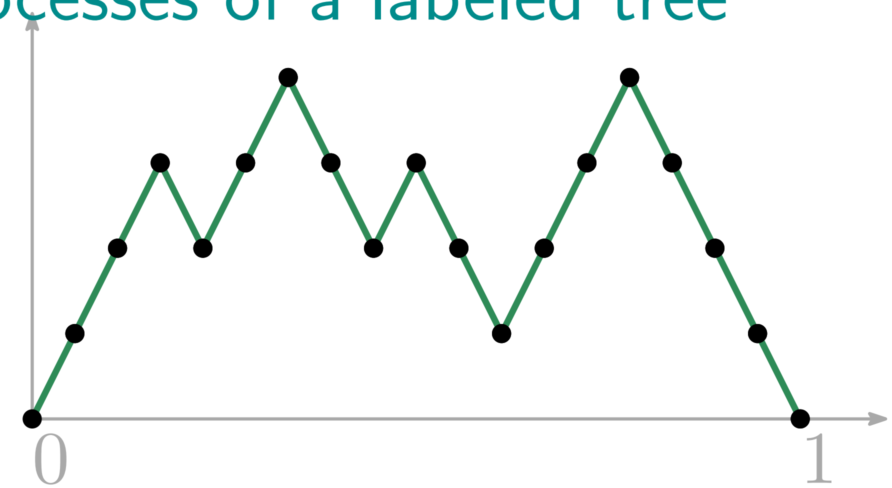

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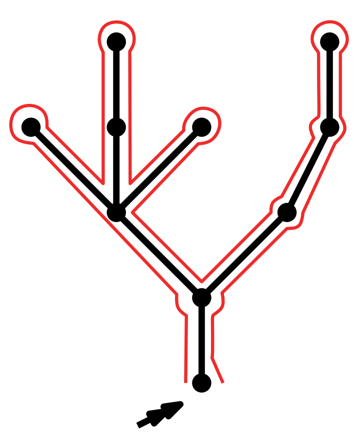

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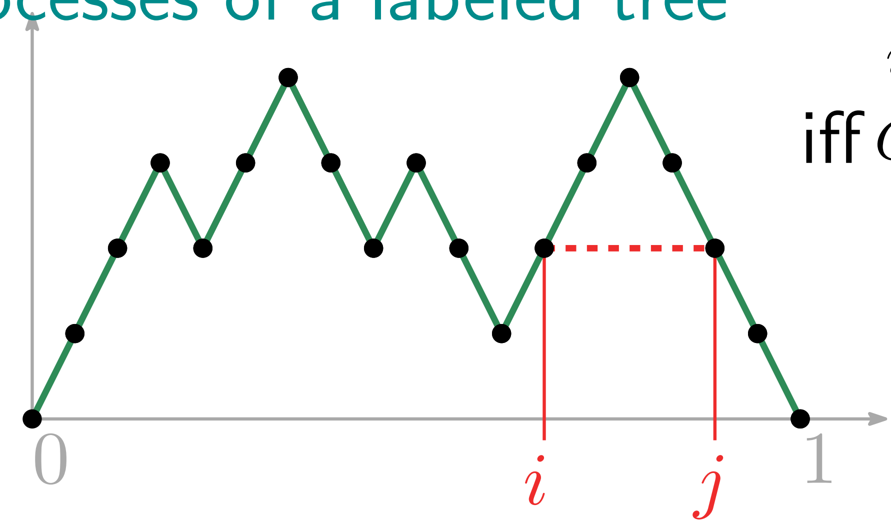

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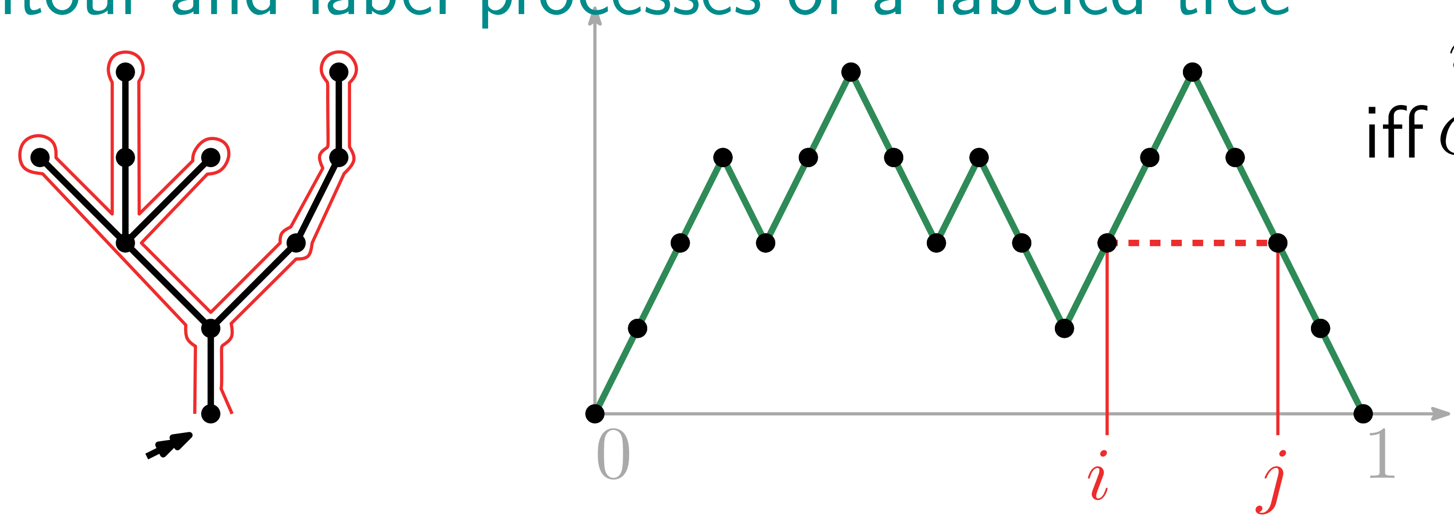

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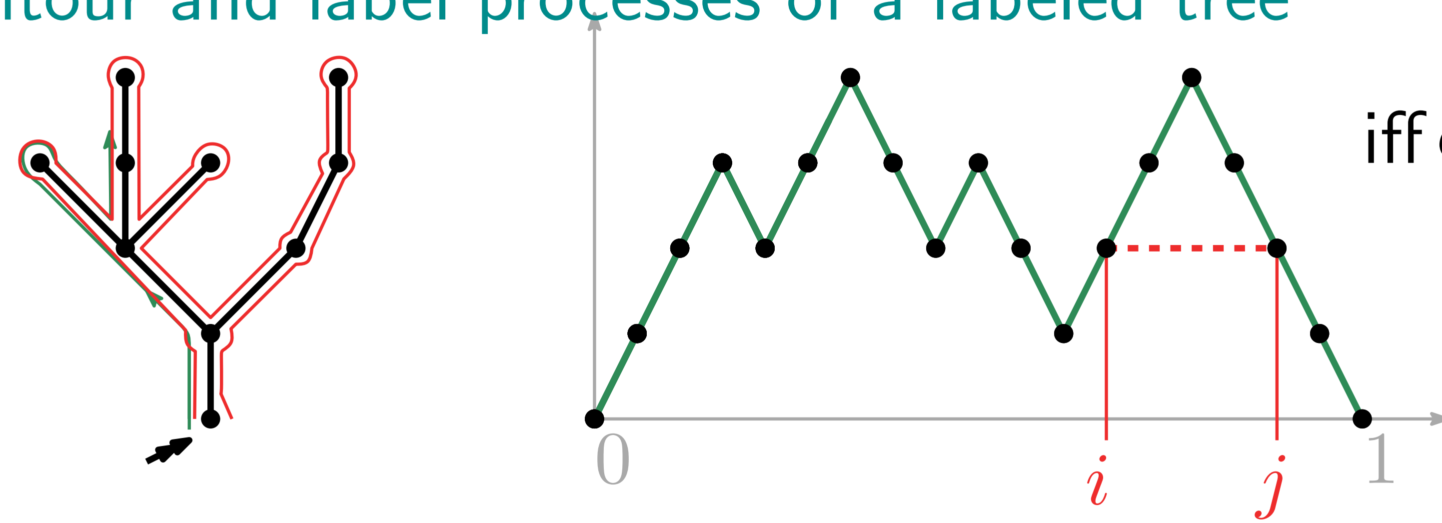

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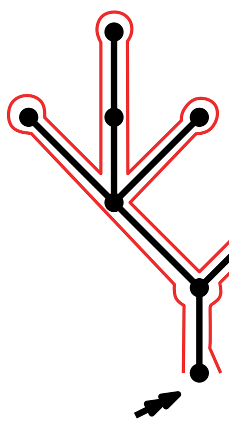
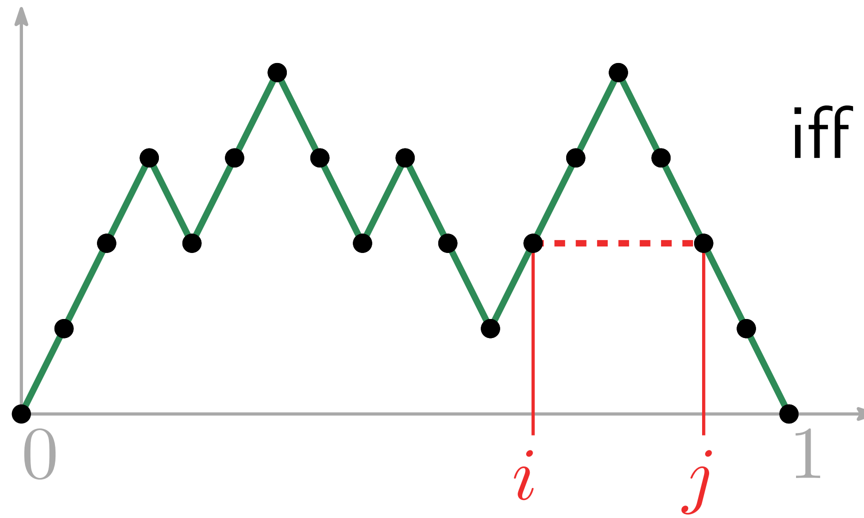
1st step : the Brownian tree [Aldous]





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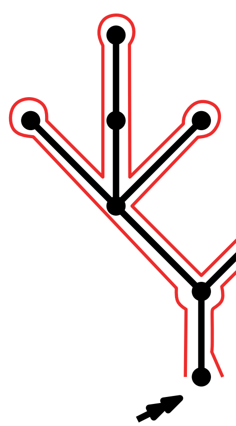

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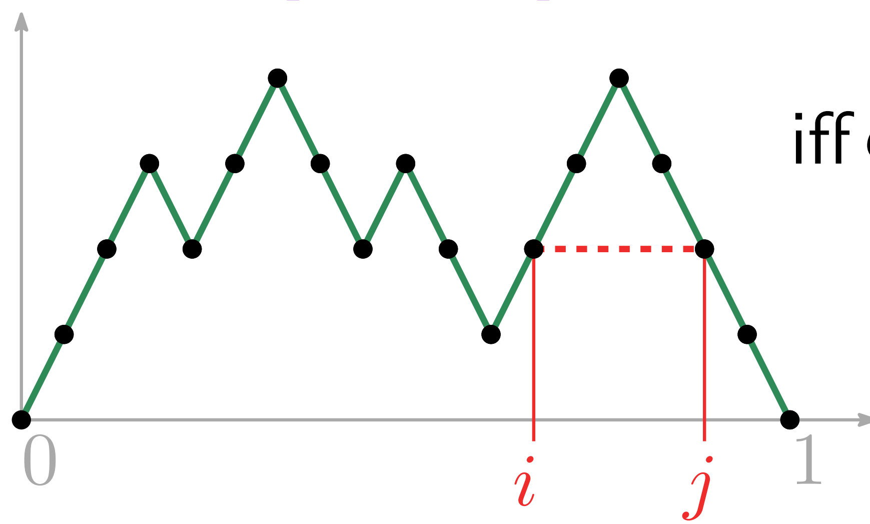

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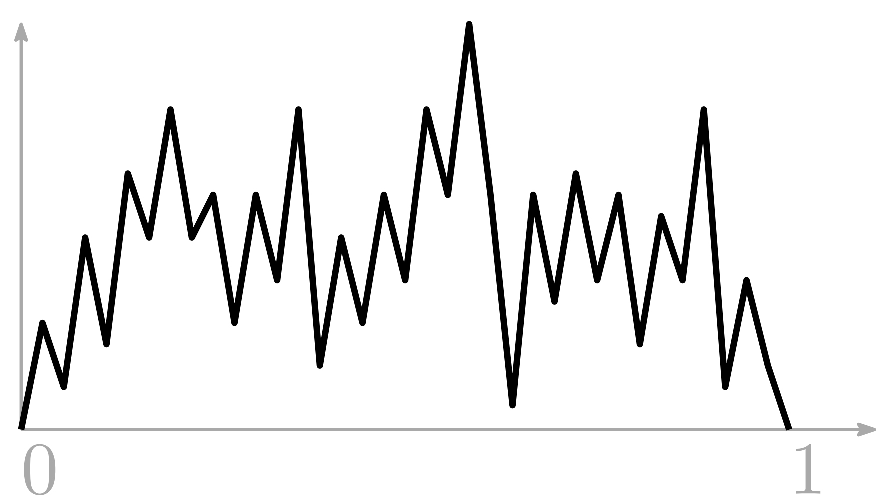

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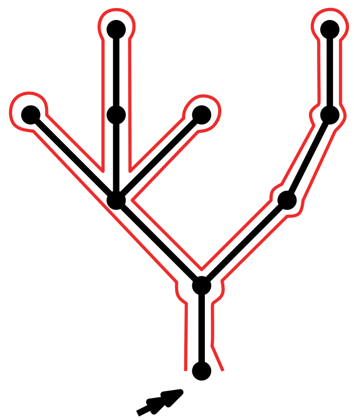

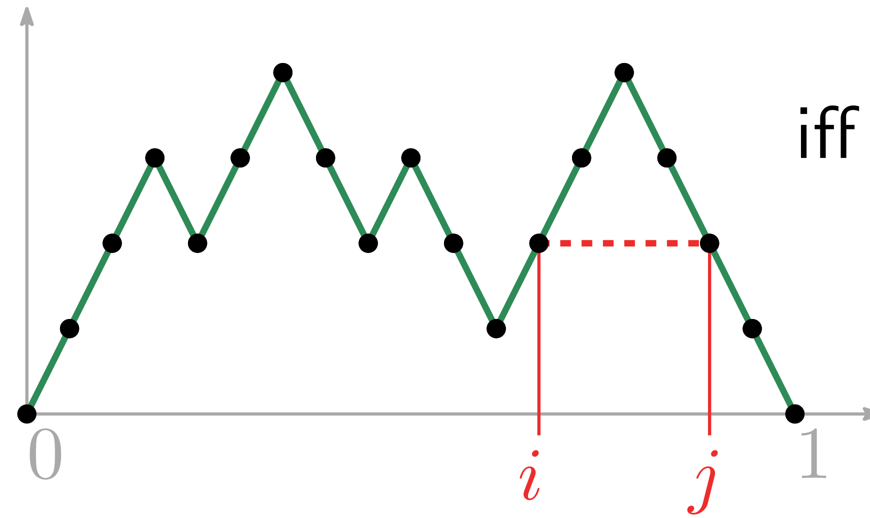

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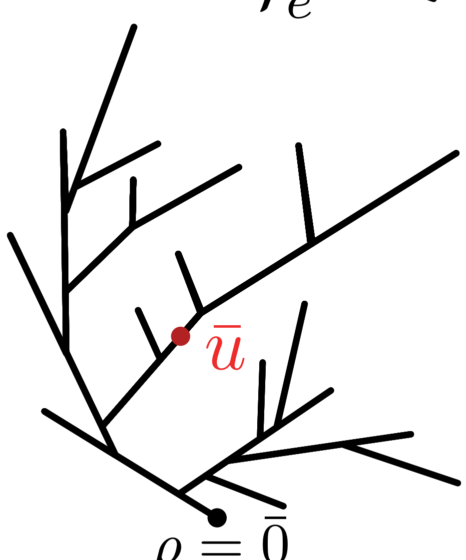

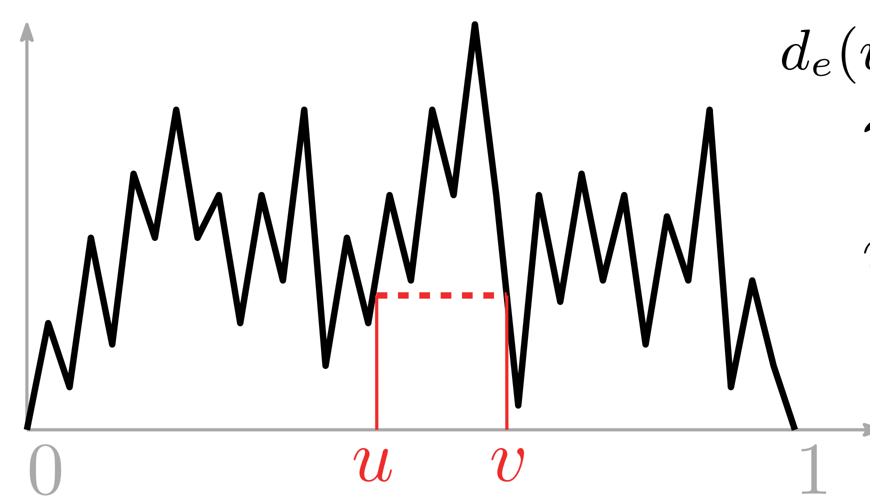

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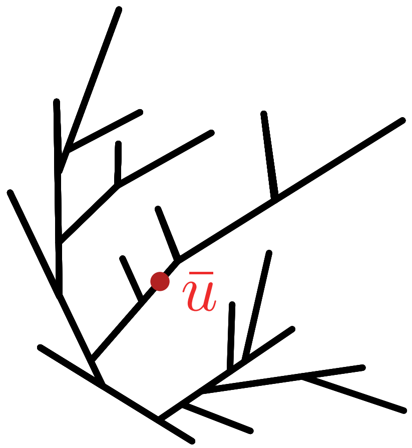

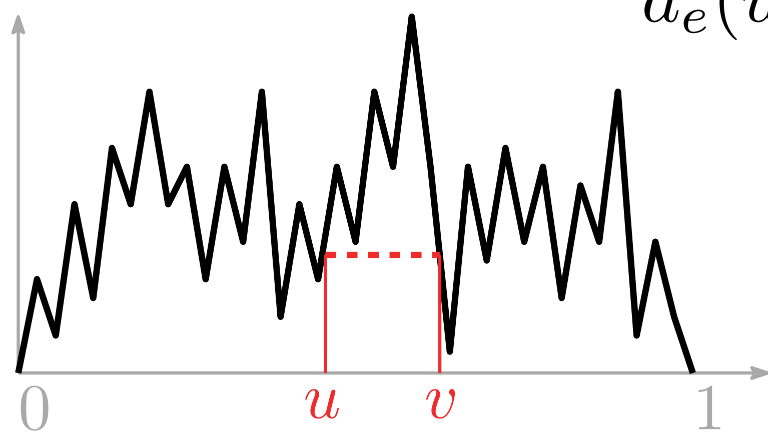

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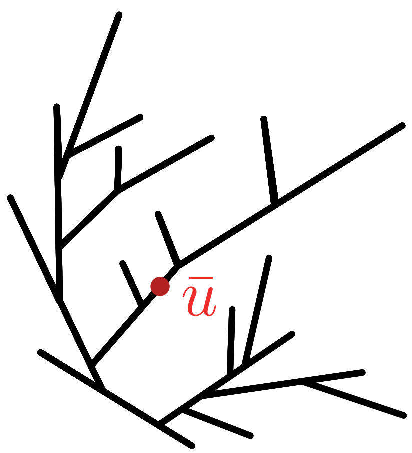

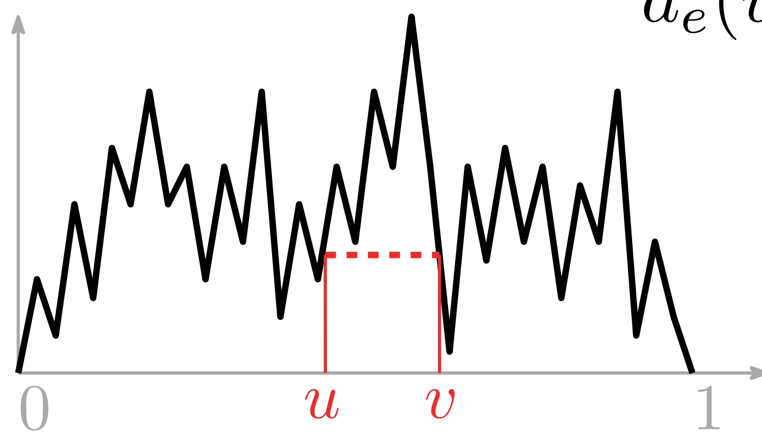

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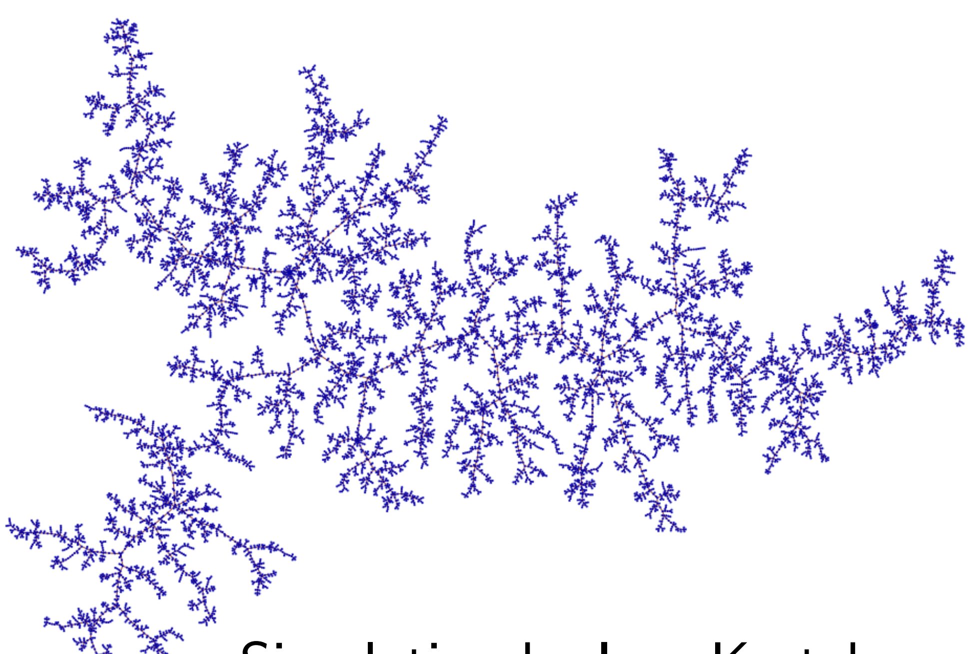
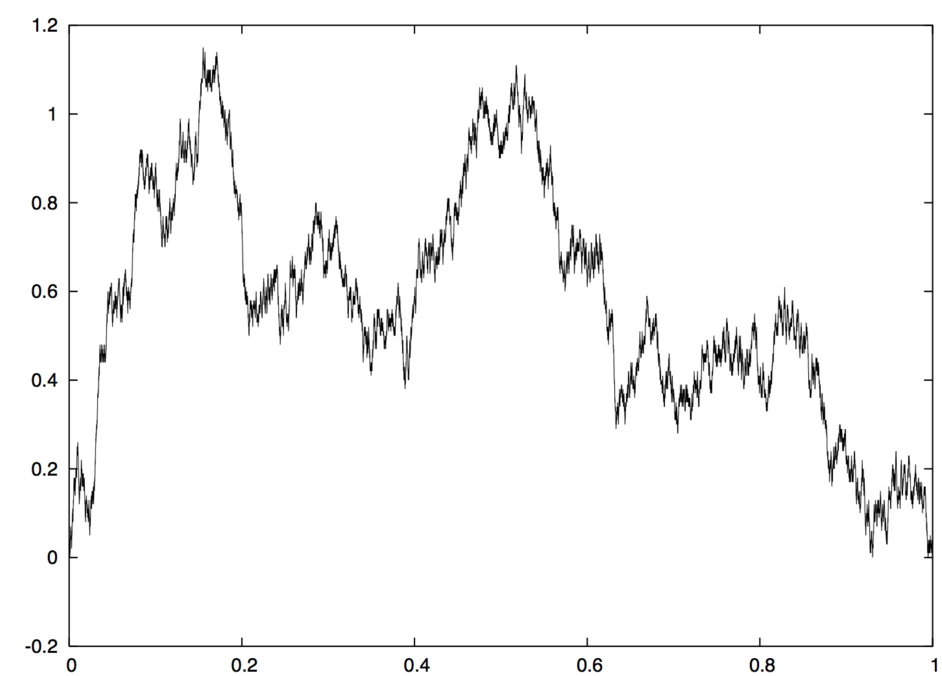


Simulation by Igor Kortchemski



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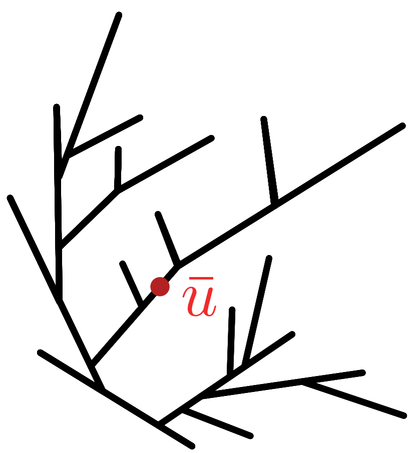

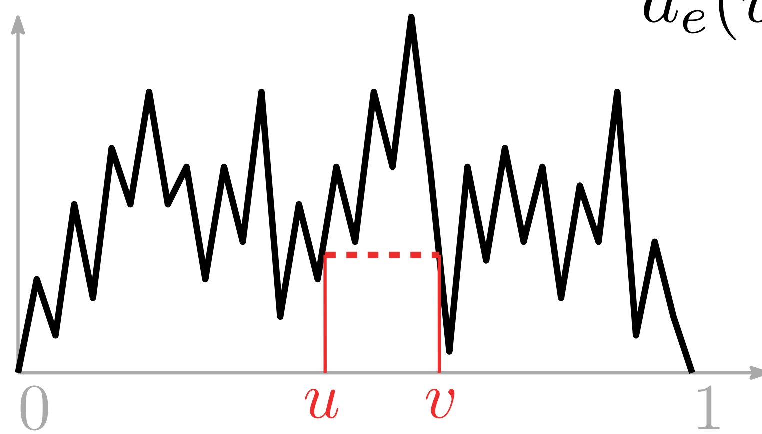

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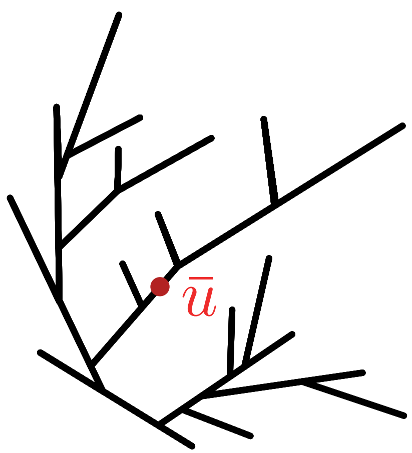
2nd step : Brownian labels

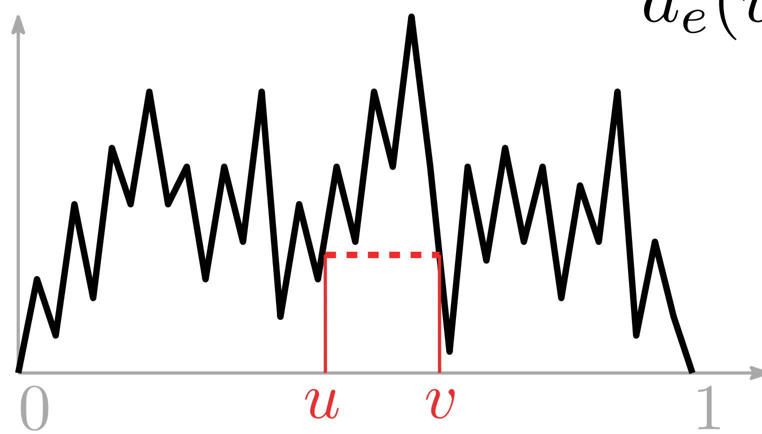
Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

$Z \sim$ Brownian motion on the tree

Brownian snake $(e_t, Z_t)_{0\leq t\leq 1}$

1st step : the Brownian tree [Aldous]





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Conditional on $\mathcal{T}_e$, Z a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_s - Z_t)^2] = d_e(s,t)$

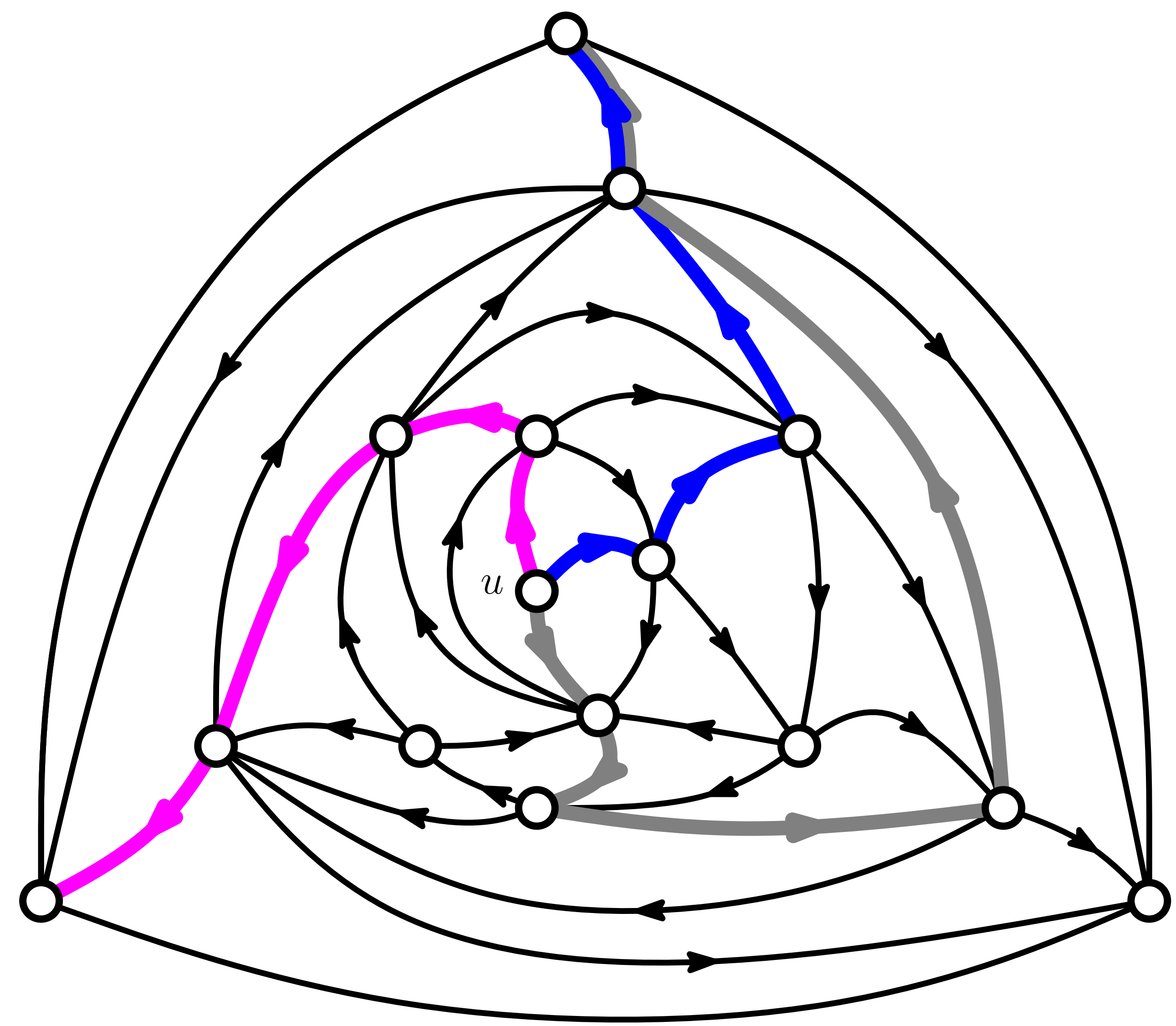
$Z \sim$ Brownian motion on the tree

Theorem : [Addario-Berry, A.]

$$\left((3n)^{-1/2} C_{\lfloor nt\rfloor}, (4n/3)^{-1/4} \tilde{Z}_{\lfloor nt\rfloor}\right)_{0\leq t\leq 1} \xrightarrow[n\to\infty]{(d)} (e_t, Z_t)_{0\leq t\leq 1},$$

Distances in simple triangulations

- Consider the **Left Most Path** from (u, v) to the root face.
- For each inner vertex : 3 LMP



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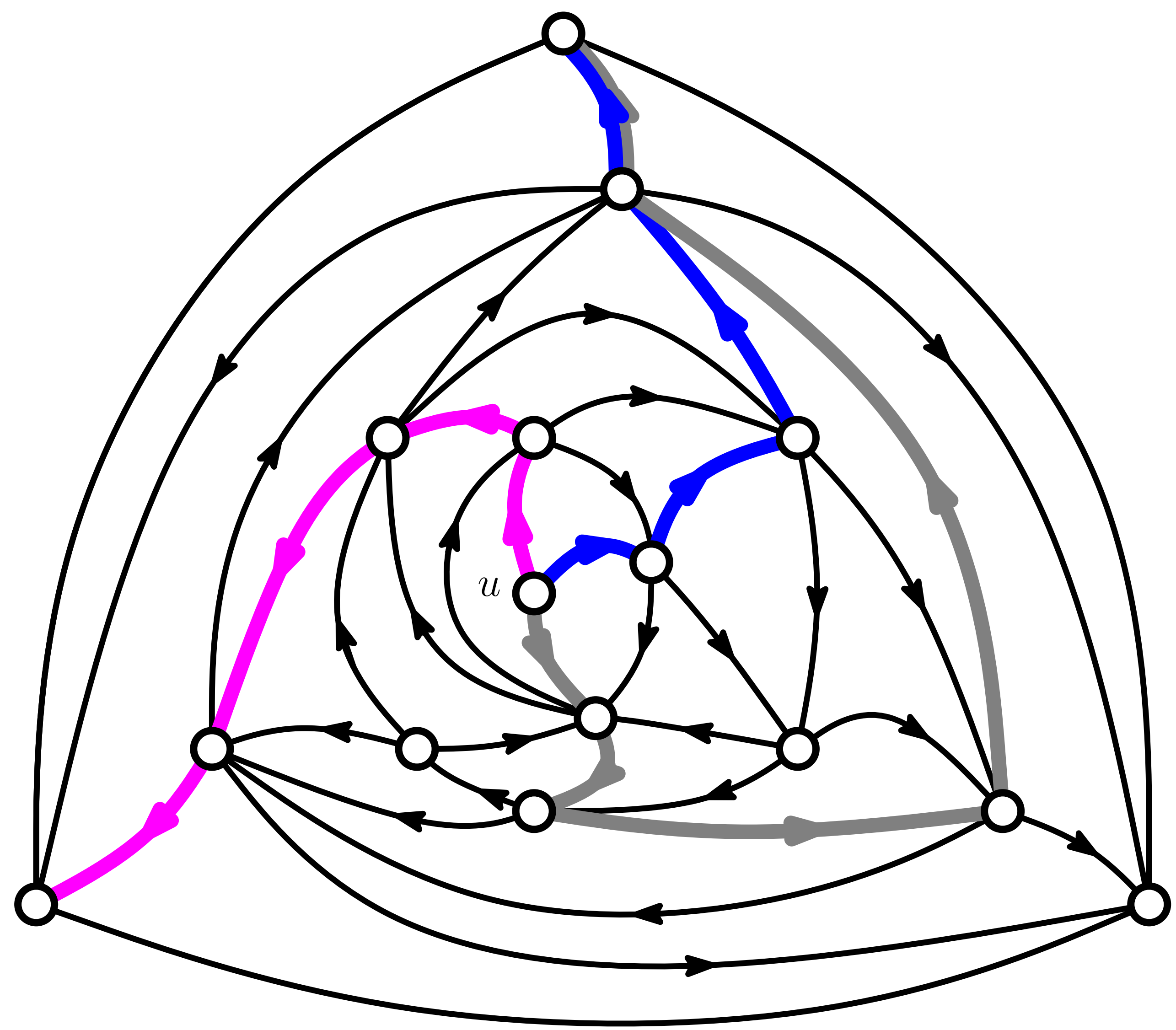

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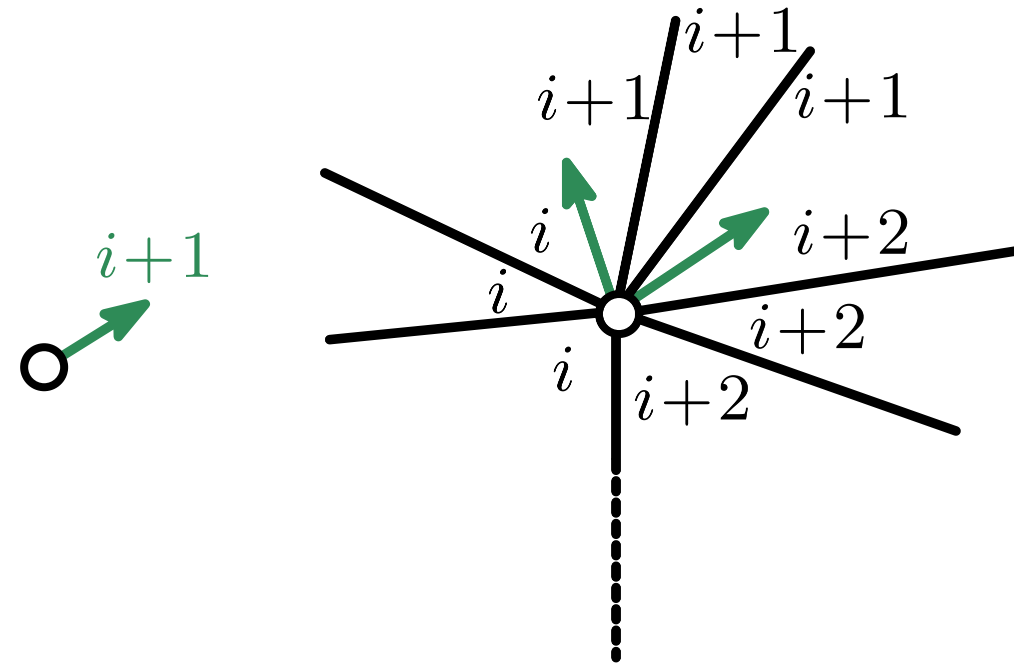

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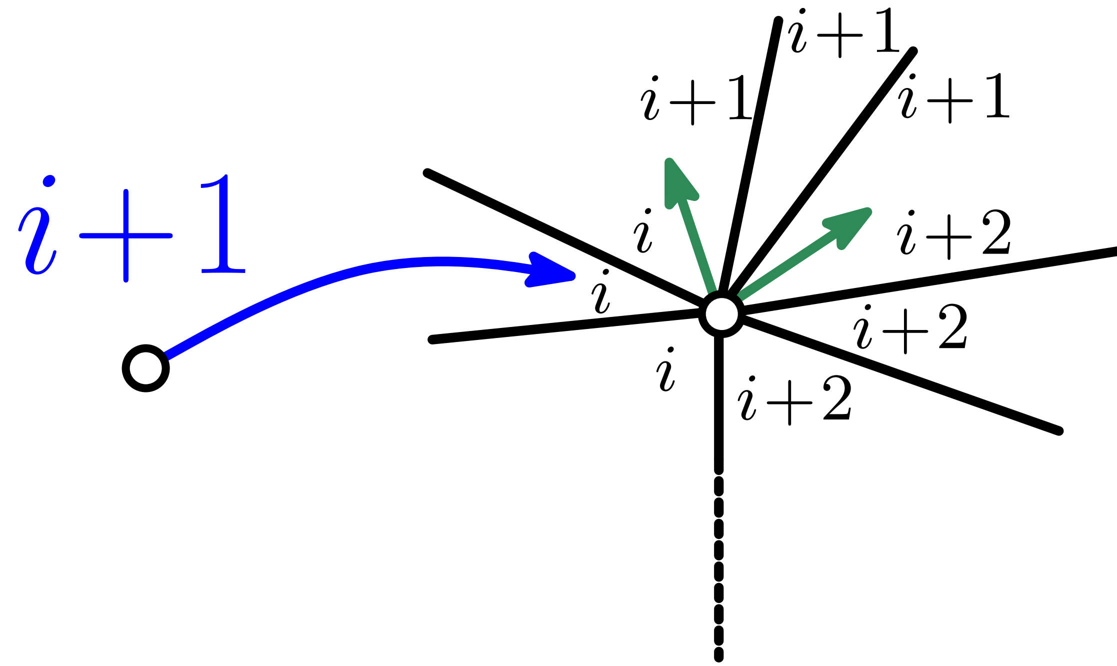

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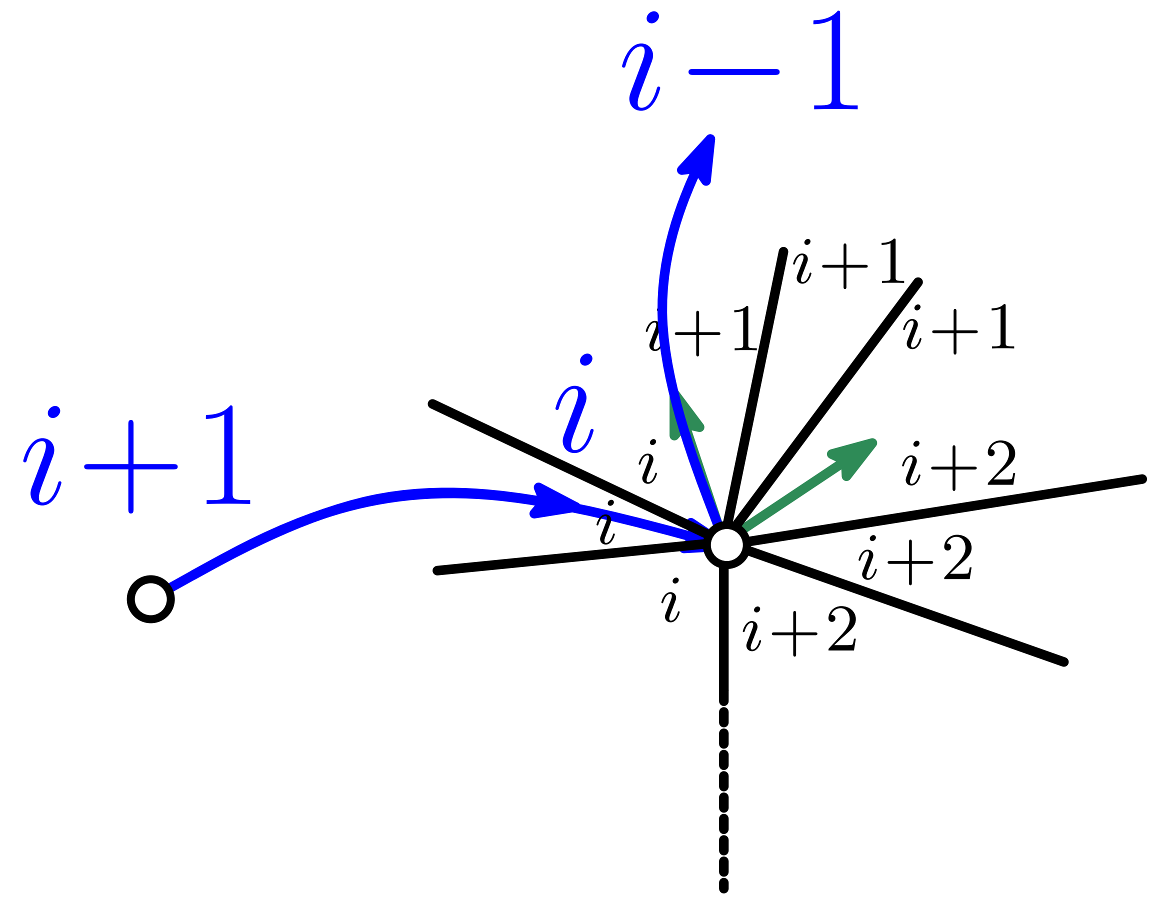
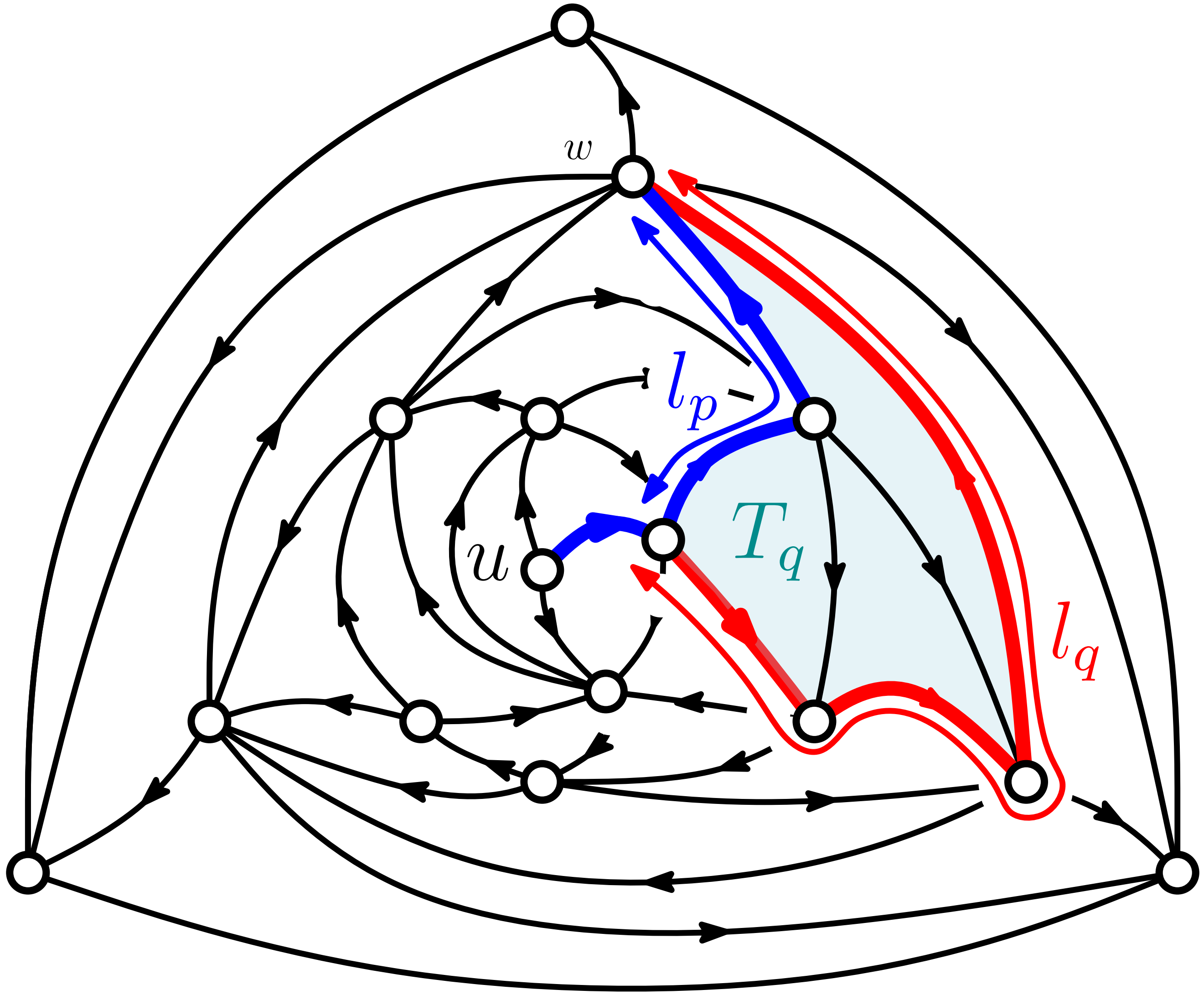


Proposition : $d_M(root, u) \leq$ Label of u

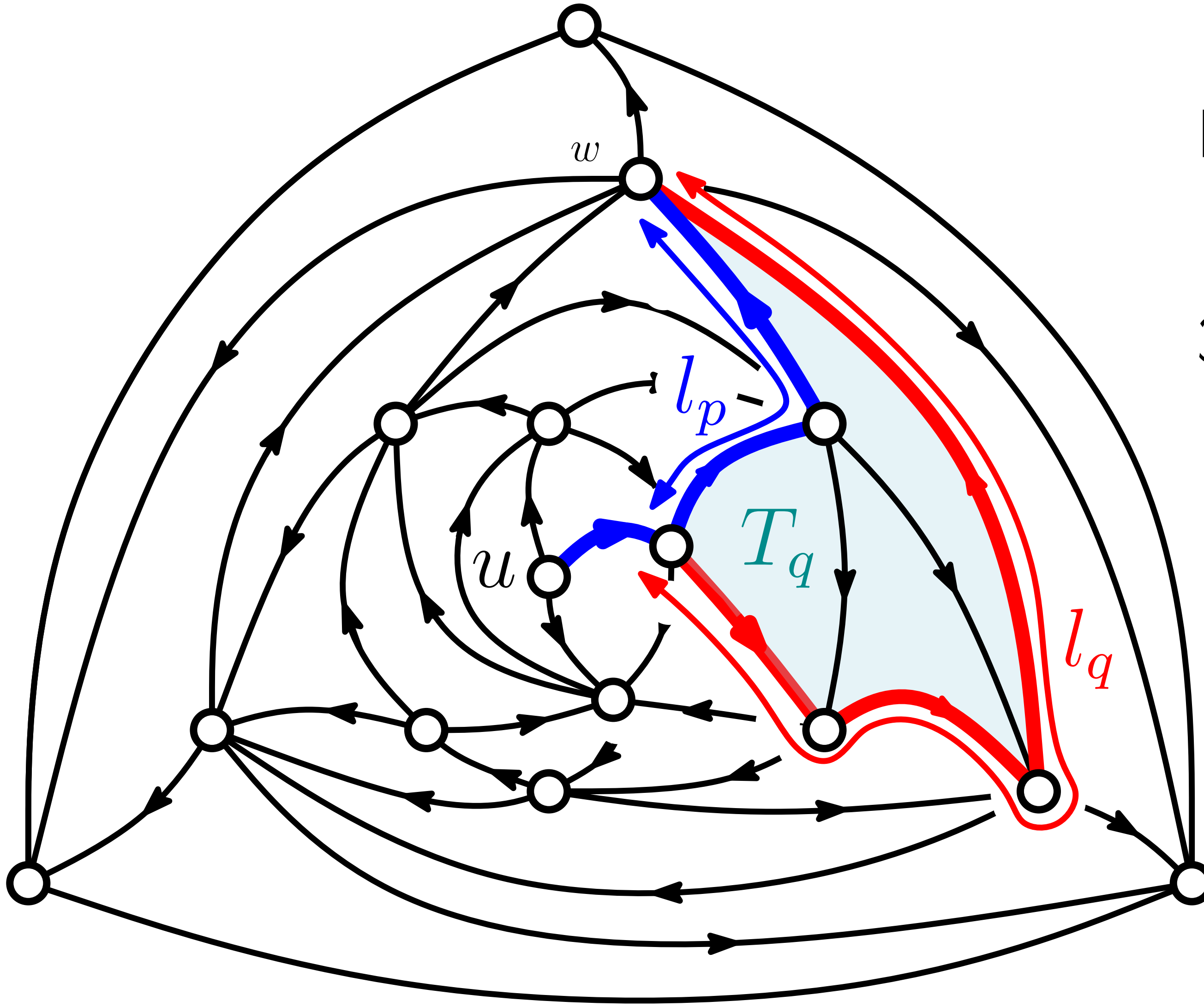
LMP are almost geodesic



Leftmost path

Another path: can it be shorter ?

LMP are almost geodesic



Euler Formula :

$|E(T_q)| = 3|V(T_q)| - 3 - (\ell_p + \ell_q)$

3-orientation + LMP :

$|E(T_q)| \geq 3|V(T_q)| - 2\ell_q - 2$

$\implies \ell_q \geq \ell_p + 1$

Leftmost path

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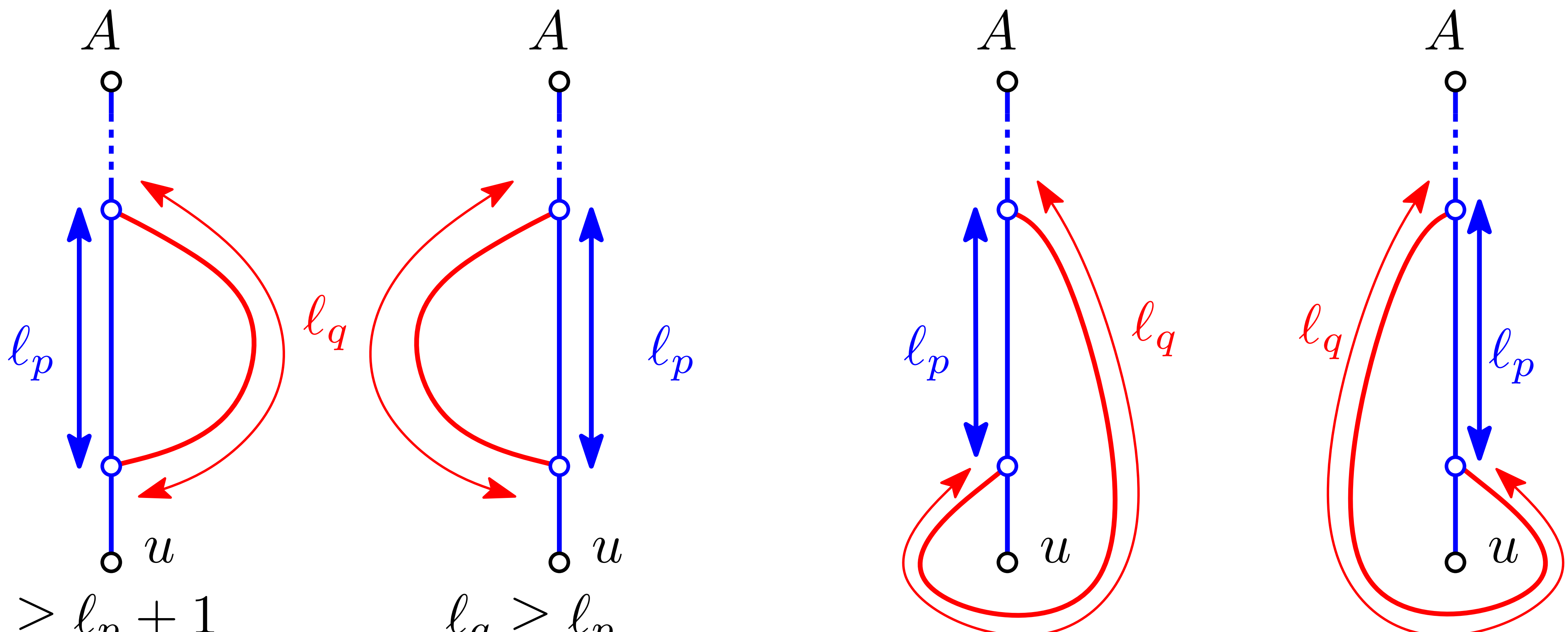
Leftmost path

Another path: can it be shorter ? YES



$\ell_q \geq \ell_p + 1$

$\ell_q \geq \ell_p$

$\ell_q \geq \ell_p + 3$

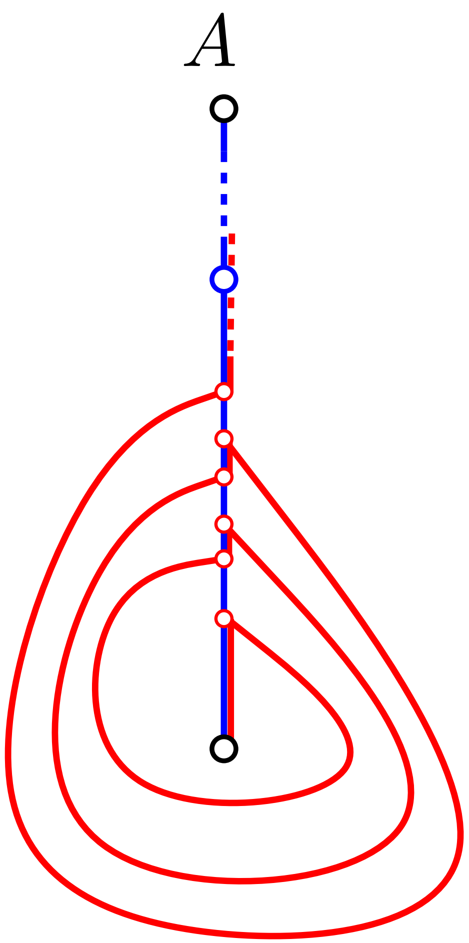
$\ell_q \geq \ell_p - 2$

with possible equality

LMP are almost geodesic

Leftmost path

Another path: can it be shorter ? YES ... but not too often



Bad configuration =
too many **windings** around the LMP

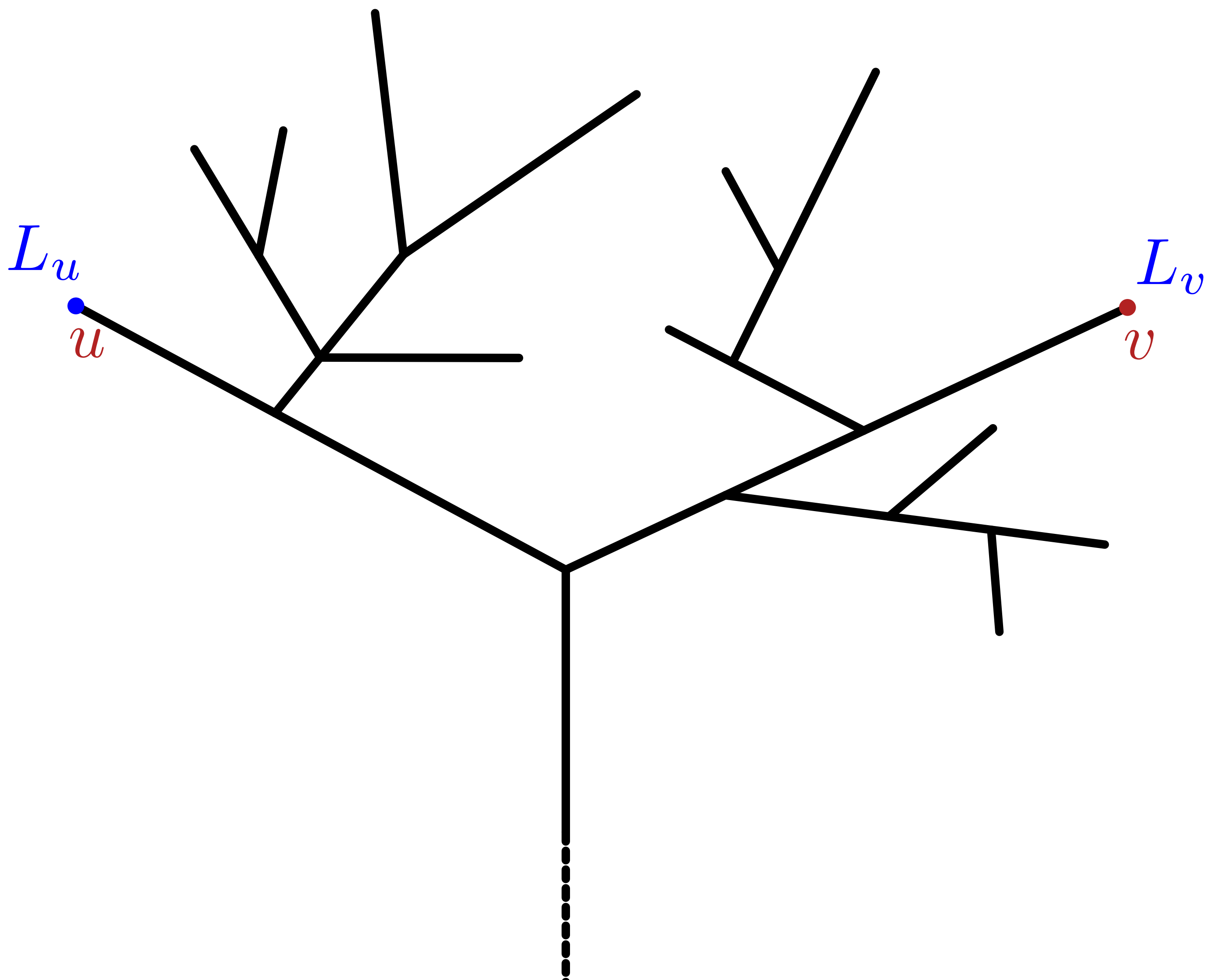
But w.h.p a winding cannot be too short.

$\Longrightarrow$ w.h.p the number of windings is $o(n^{1/4})$.

Proposition:
For $\varepsilon > 0$, let $A_{n,\varepsilon}$ be the event that there exists $u \in M_n$ such that
Label of $u \geq d_{M_n}(u, root) + \varepsilon n^{1/4}$.
Then under the uniform law on $\mathcal{M}_n$, for all $\varepsilon > 0$:

$$\mathbb{P}(A_{n,\varepsilon}) \to 0.$$

Distances are tight



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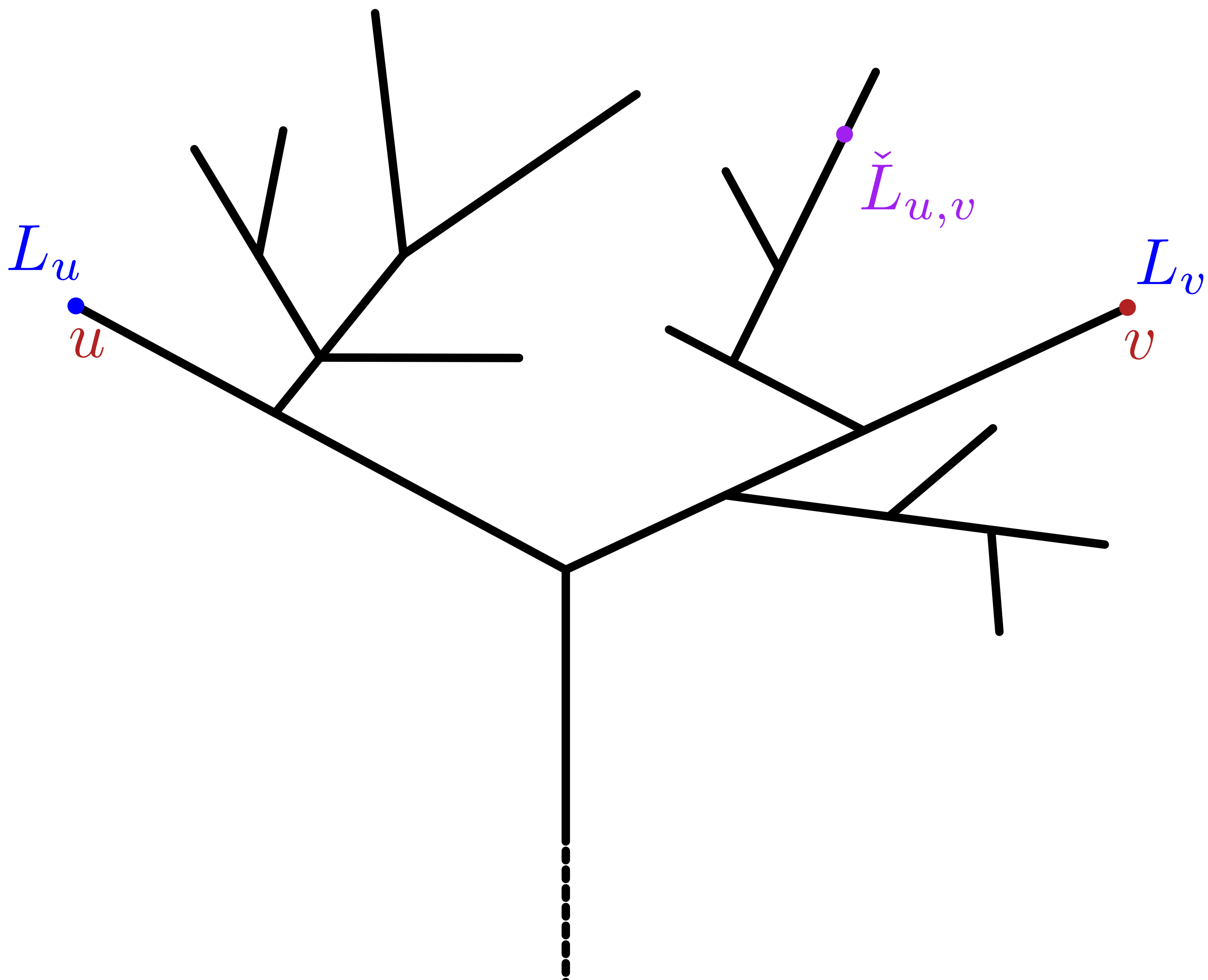

$$\check{L}_{u,v} = \min\{L_s, u \leq s \leq v\}$$

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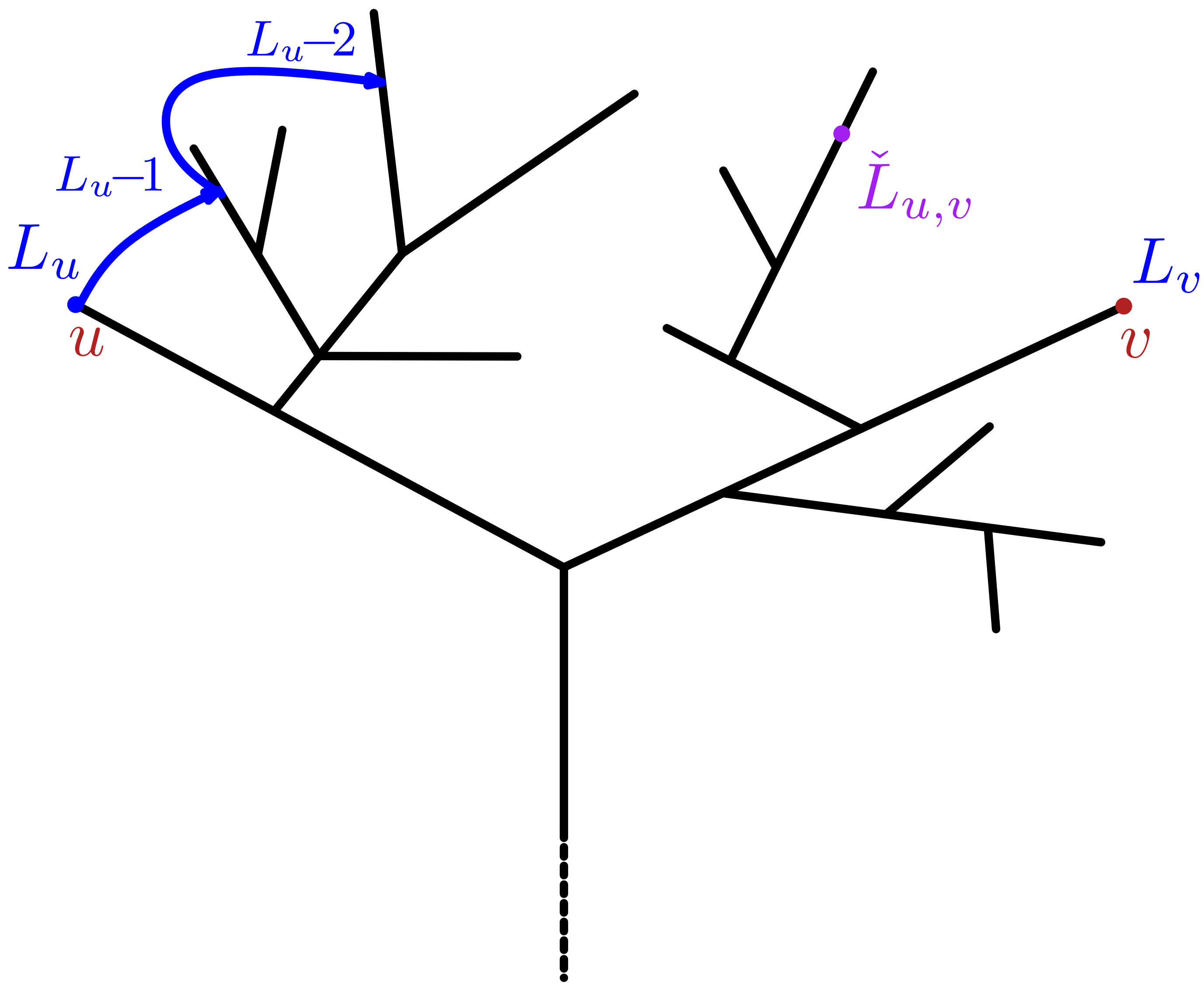

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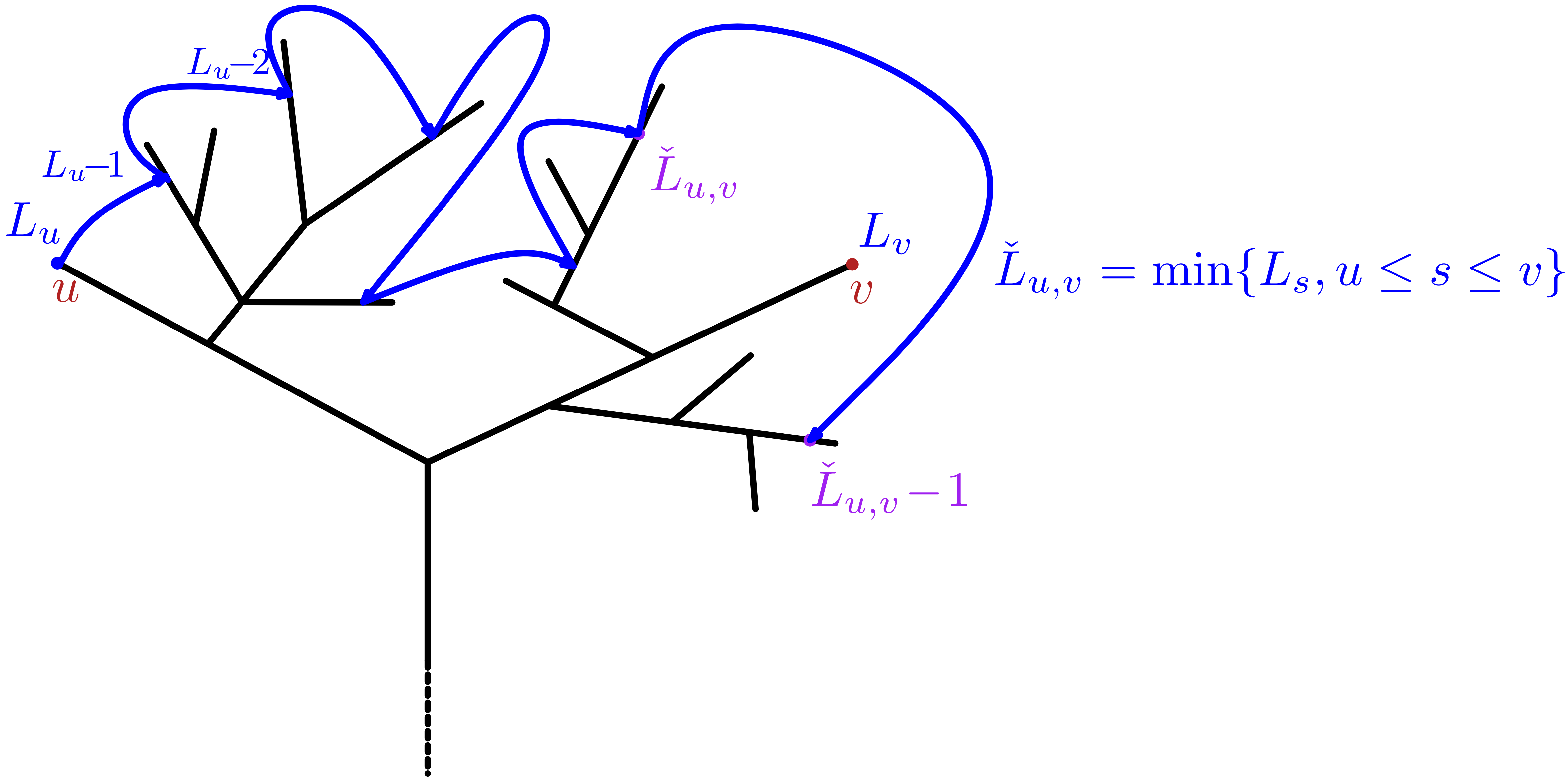

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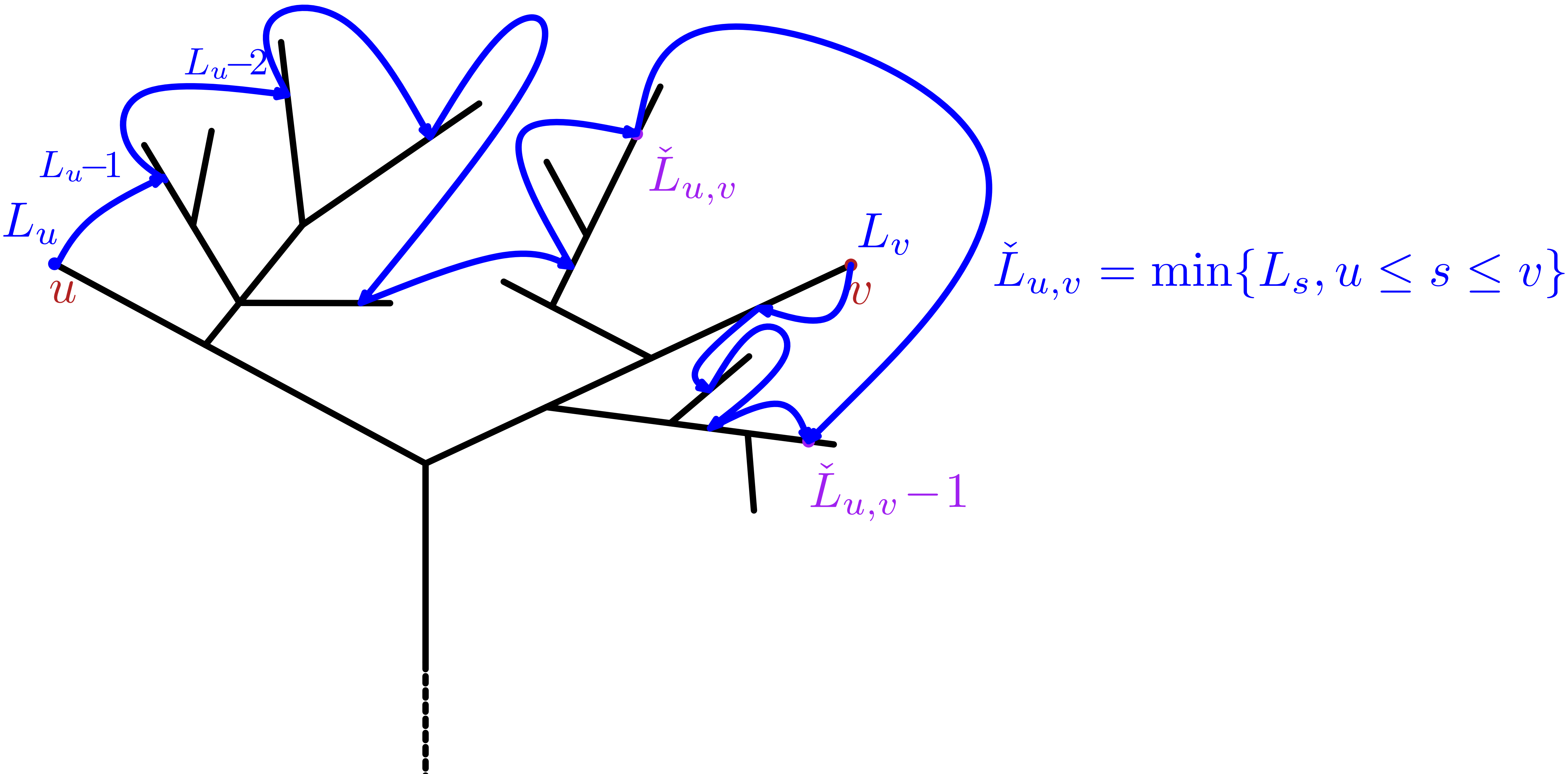

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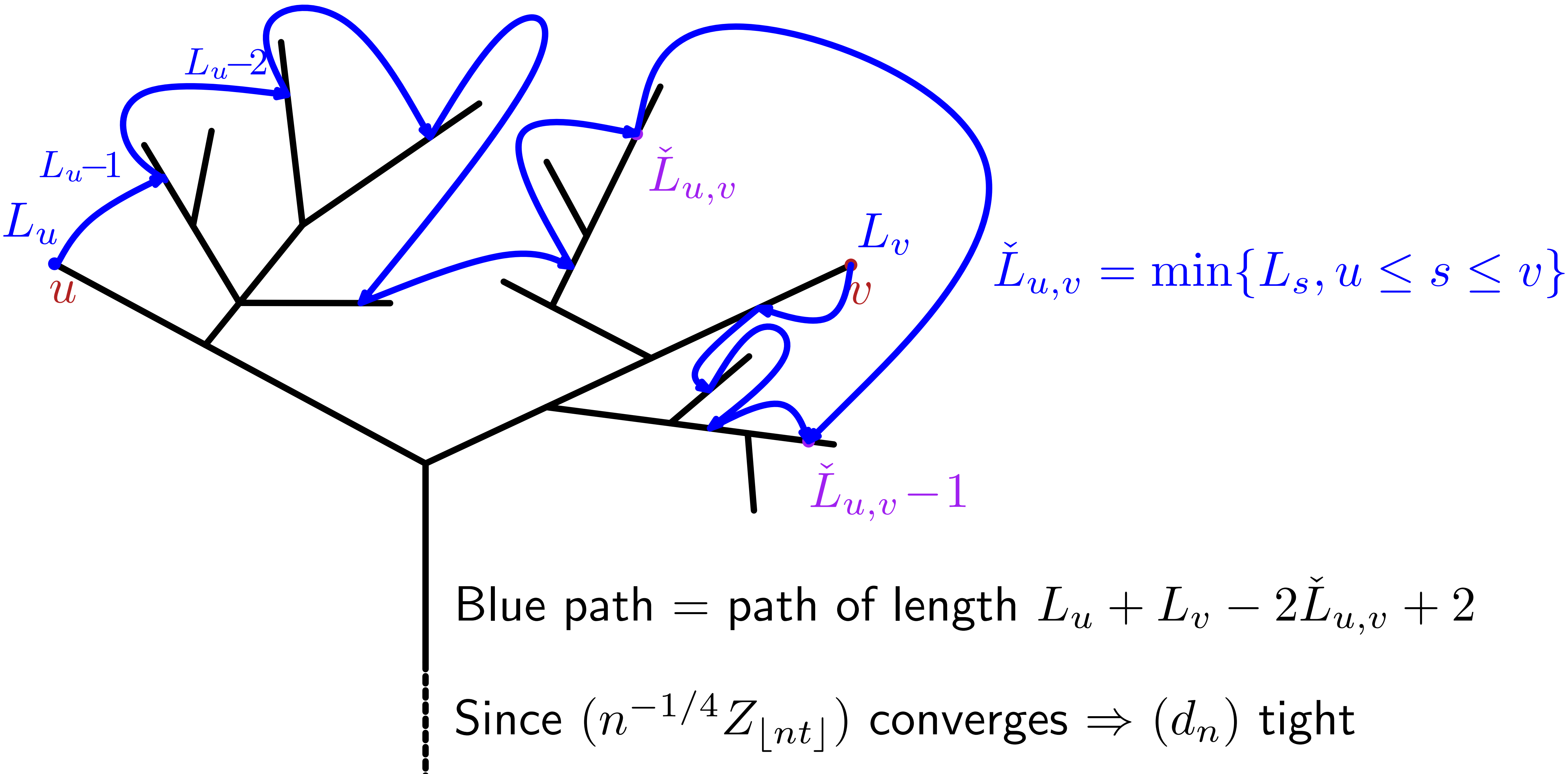
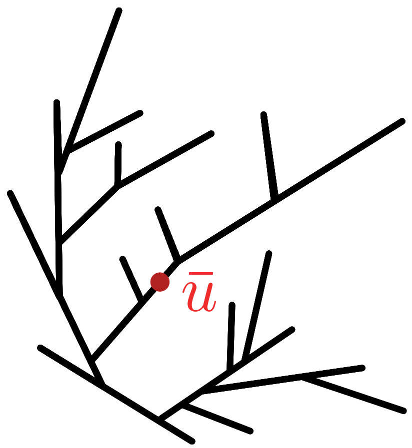
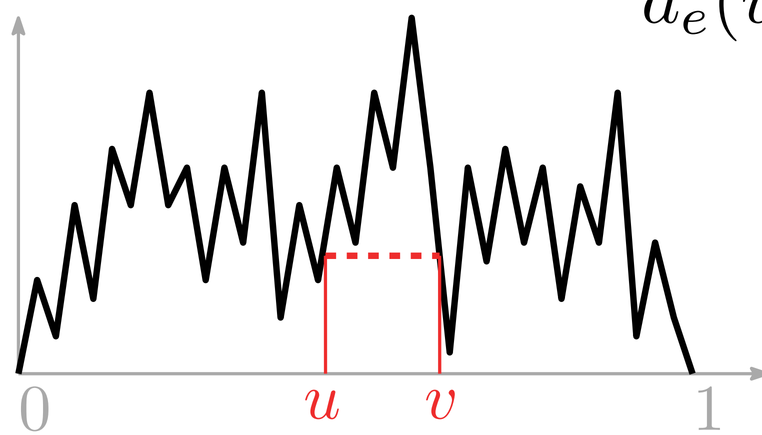


Blue path = path of length $L_u + L_v - 2\check{L}_{u,v} + 2$

Since $(n^{-1/4} Z_{\lfloor nt \rfloor})$ converges $\Rightarrow$ (d_n) tight

The Brownian map





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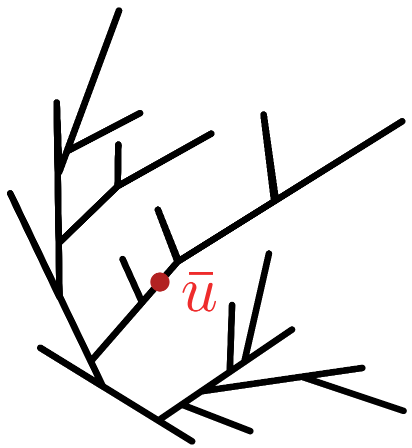

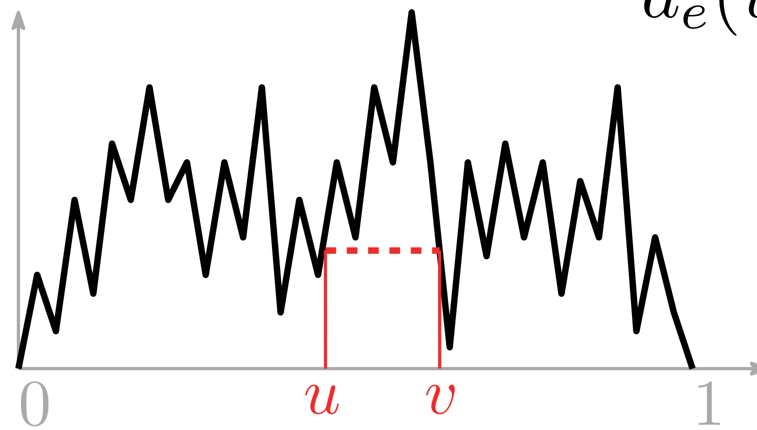

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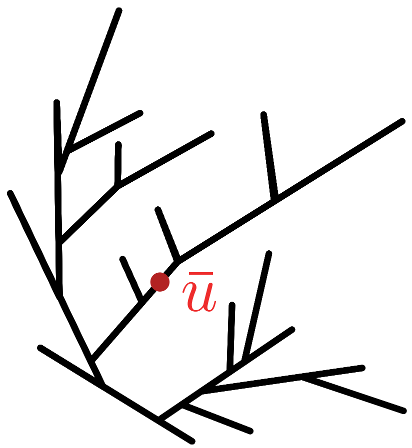

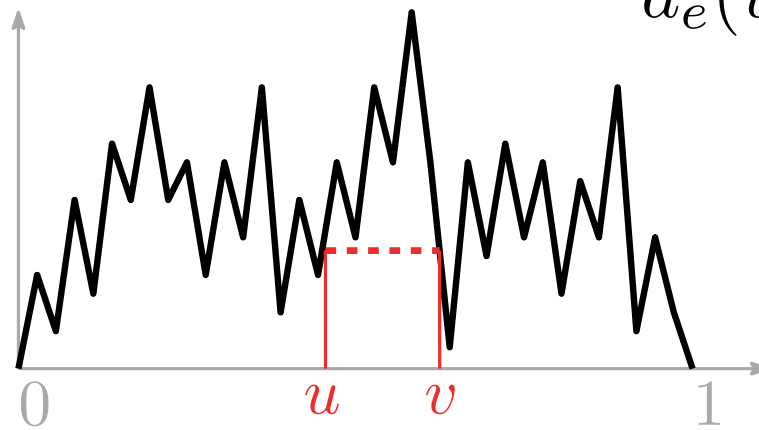

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$$D^*(a,b) = \inf\Big\{\sum_{i=1}^{k-1} D^\circ(a_i, a_{i+1}) : k \ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b\Big\},$$

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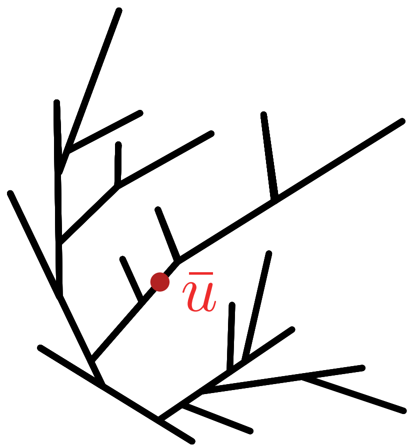

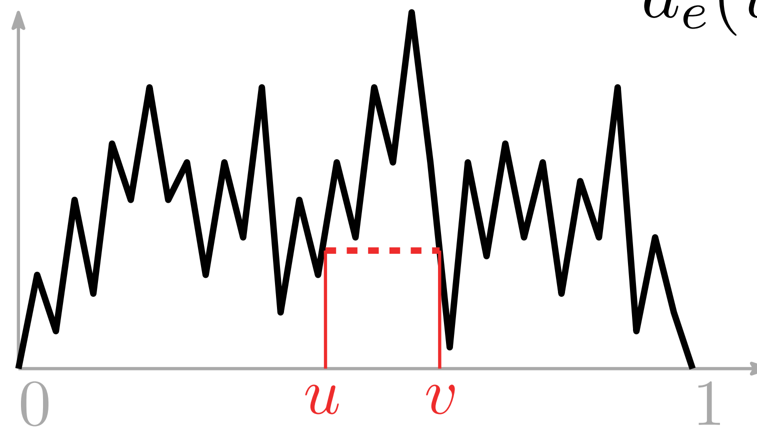

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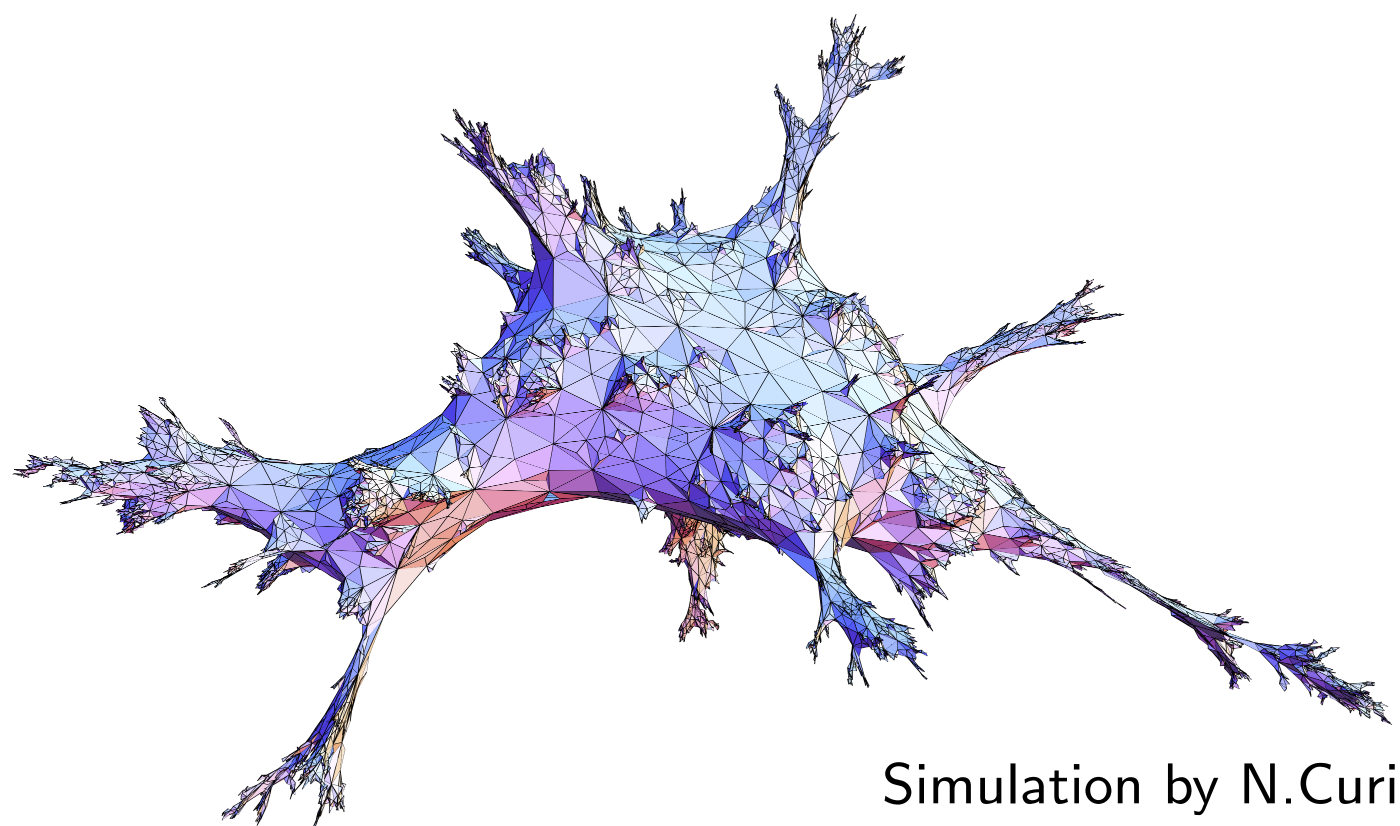
Then $M = (\mathcal{T}_e/\sim_{D^\star}, D^*)$ is the **Brownian map**.

The result

Theorem : [Addario-Berry, A.]
(M_n) = sequence of random **simple** triangulations, then:

$$\left(M_n, \left(\frac{3}{4n}\right)^{1/4} d_{M_n}\right) \xrightarrow{(d)} \text{Brownian map, for the GH distance}$$



Simulation by N.Curien

Beyond the universality

Simple triangulations converge to the Brownian map
$\Rightarrow$ properties of the Brownian map from the simple triangulations ?

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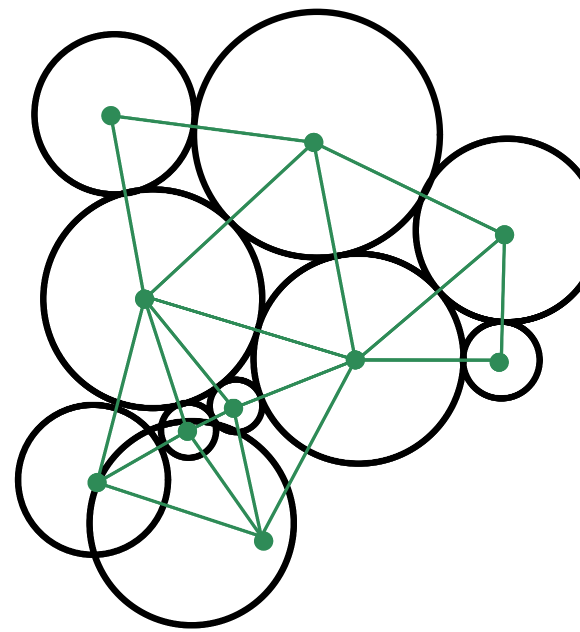
$\Rightarrow$ properties of the Brownian map from the simple triangulations ?

One motivation : Circle-packing theorem

Each simple triangulation M has a unique (up to Möbius transformations and reflections) circle packing whose tangency graph is M.
[Koebe-Andreev-Thurston]

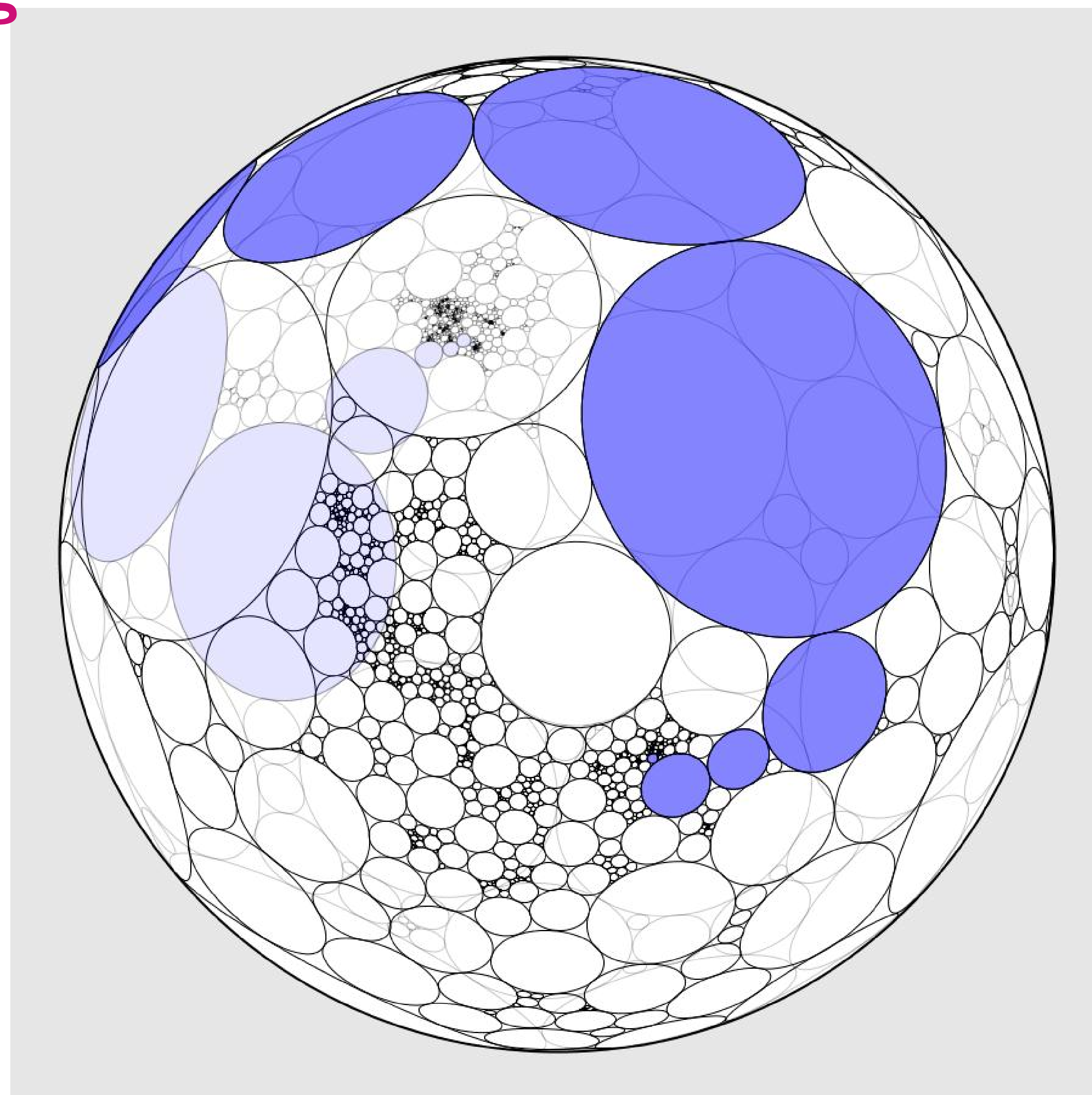


Gives a canonical embedding of simple triangulations in the sphere and possibly of their limit.

Random circle packing

Random circle packing = canonical embedding of random simple triangulation in the sphere.

Gives a way to define a canonical embedding of their limit ?



Team effort : code by Kenneth Stephenson, Eric Fusy and our own.

Perspectives

Same approach works also for simple quadrangulations, and for simple maps (ongoing work with Bernardi, Collet, Fusy).

Can we make this approach work for the general setting of bijections developped in [A.,Poulalhon] and in [Bernardi, Fusy] ?

Can we use this result to get a glimpse about the conformal structure of the Brownian map ?

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Thank you !

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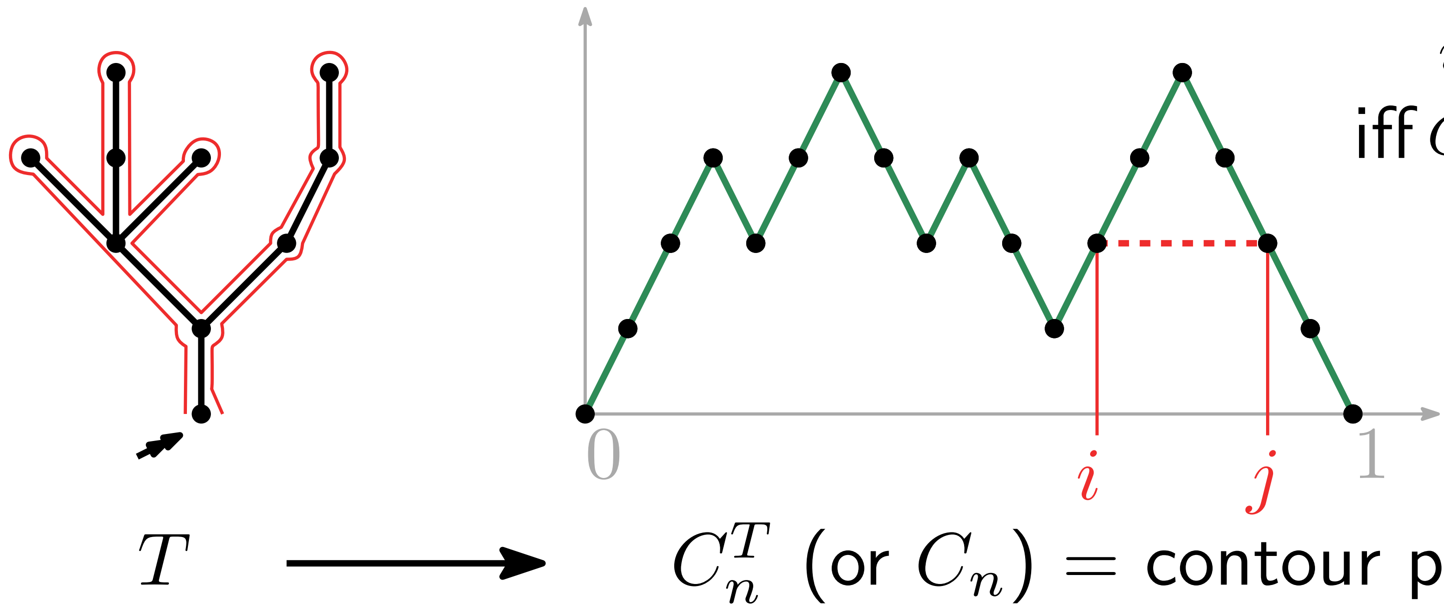
1st step : the Brownian tree [Aldous]



i and j = same vertex of T
iff $C_n(i) = C_n(j) = \min_{i\leq k\leq j} C_n(k)$

C_n^T (or C_n) = contour process

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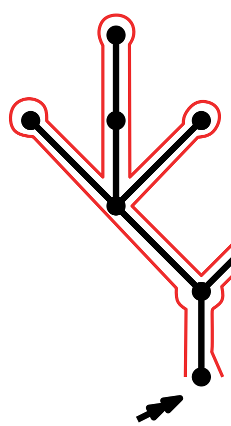

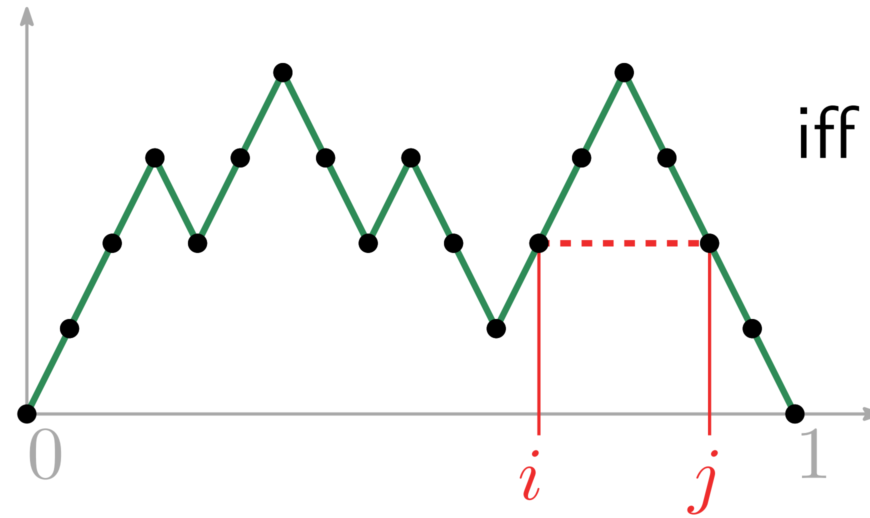

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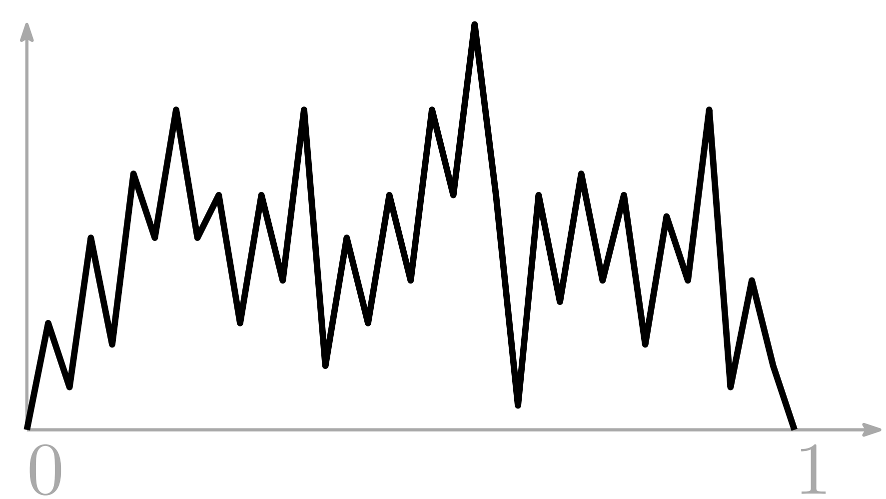

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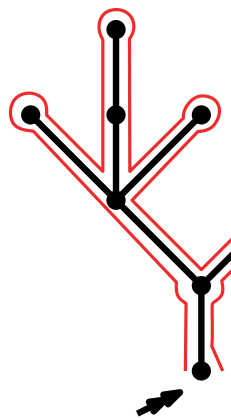

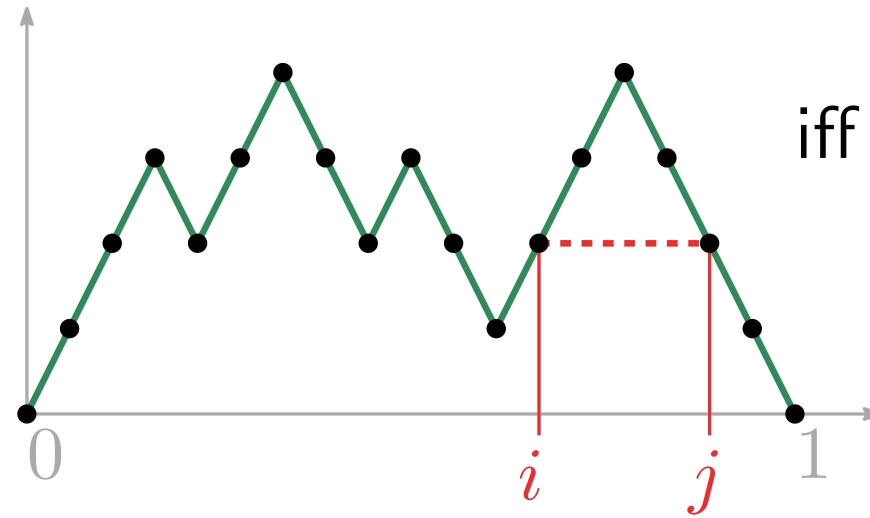

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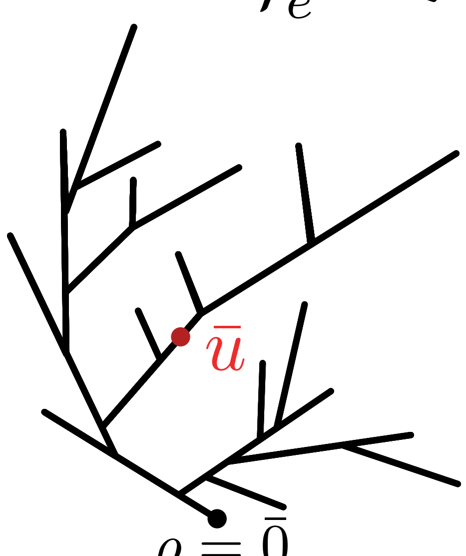

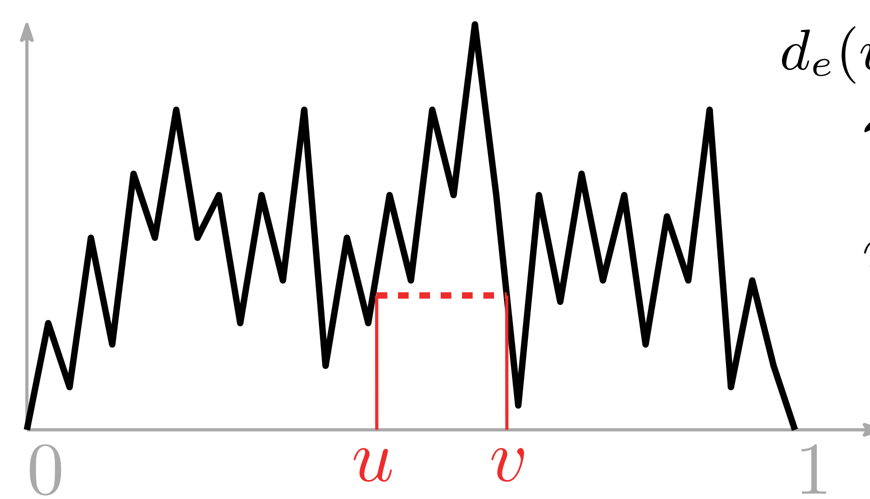

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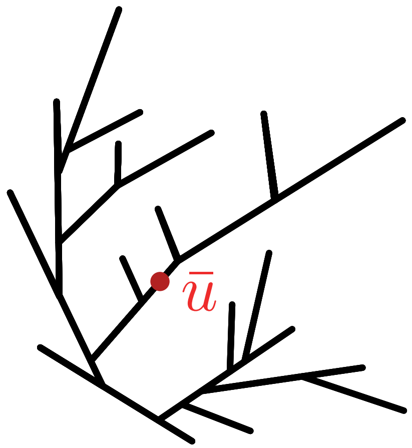

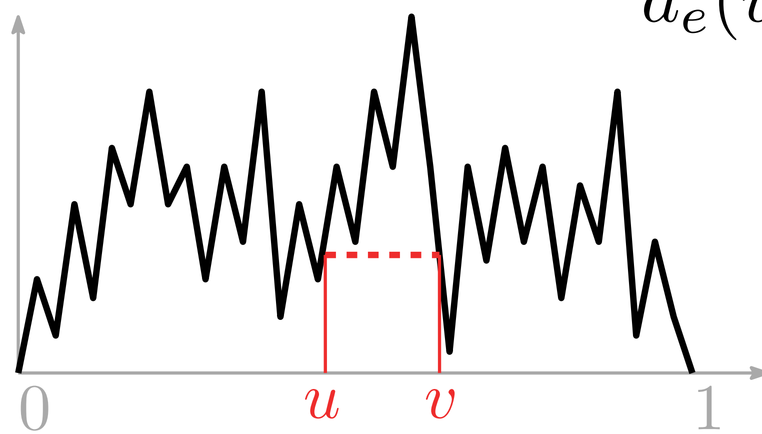

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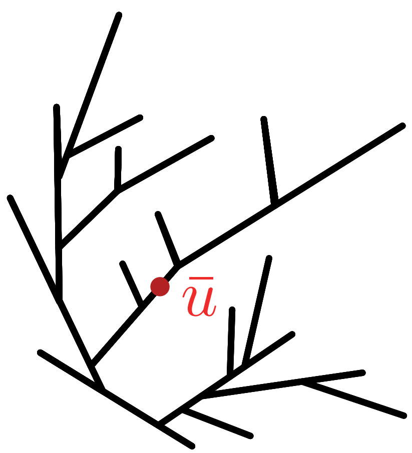

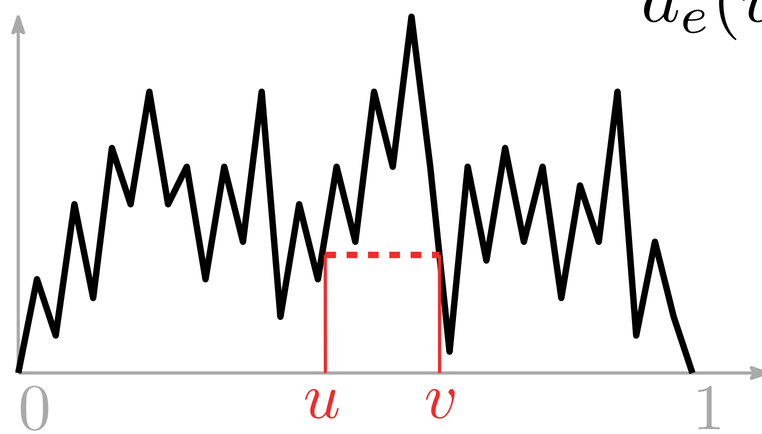

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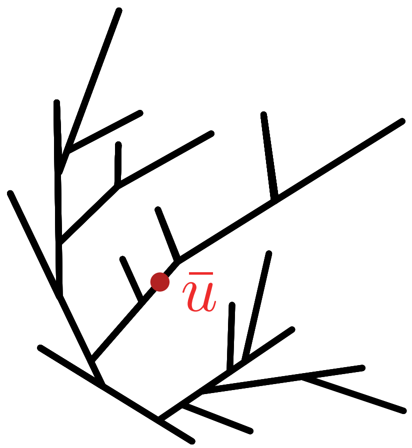
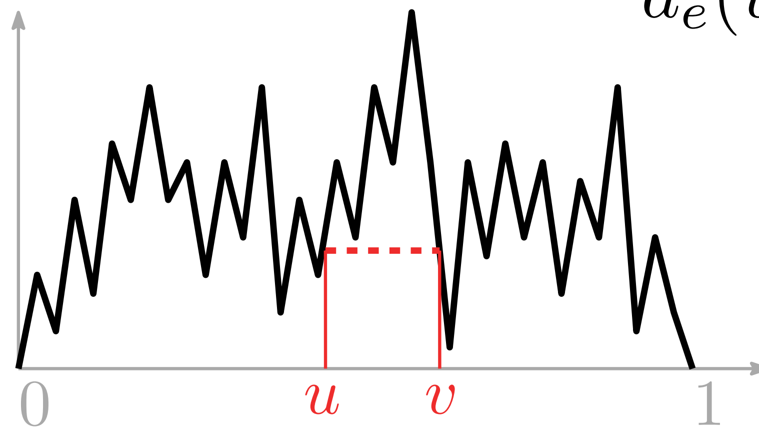
The Brownian map





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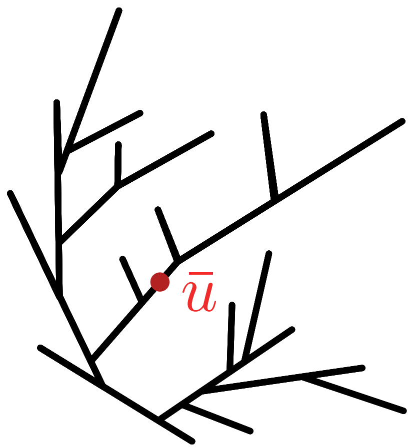

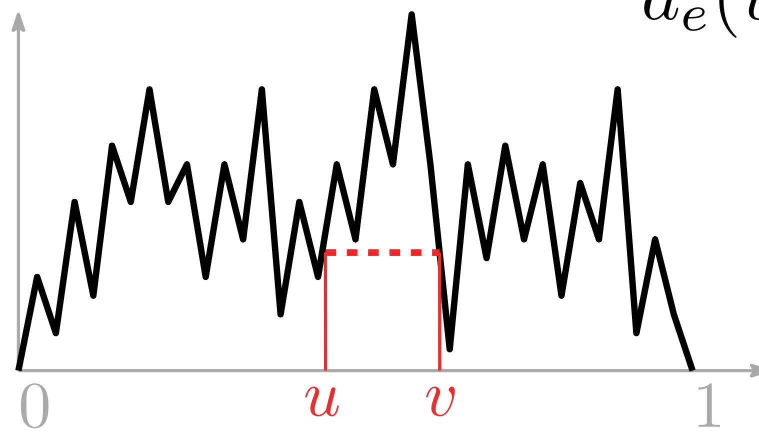

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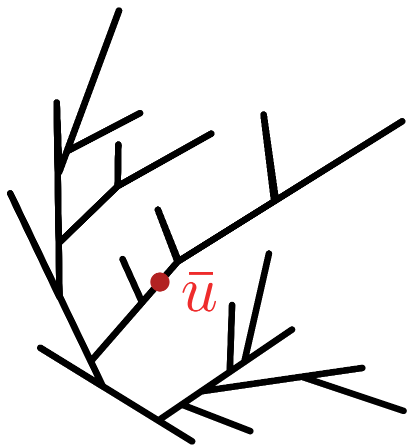
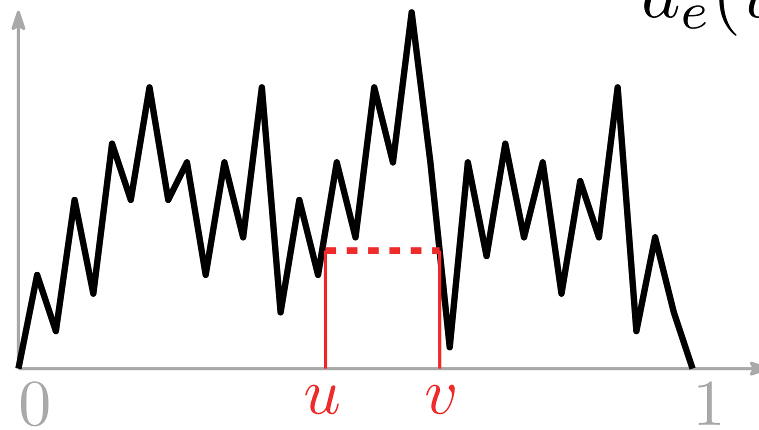
$Z \sim$ Brownian motion on the tree

$$D^\circ(s,t) = Z_s + Z_t - 2\max\left(\inf_{s\le u\le t} Z_u, \inf_{t\le u\le s} Z_u\right), \quad s,t \in [0,1]\,.$$

The Brownian map





$d_e(u,v) = e_u + e_v - 2\min_{u \le s \le v} e_s$

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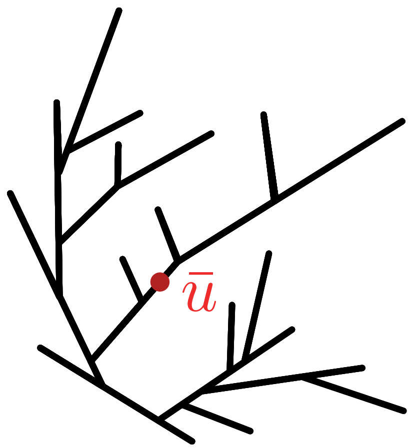
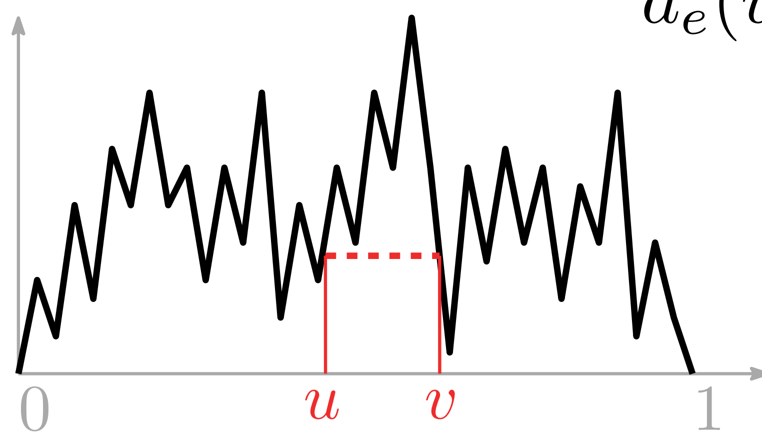
$$D^*(a,b) = \inf\left\{\sum_{i=1}^{k-1} D^\circ(a_i,a_{i+1}) : k\ge 1, a = a_1, a_2, \ldots, a_{k-1}, a_k = b\right\},$$

The Brownian map





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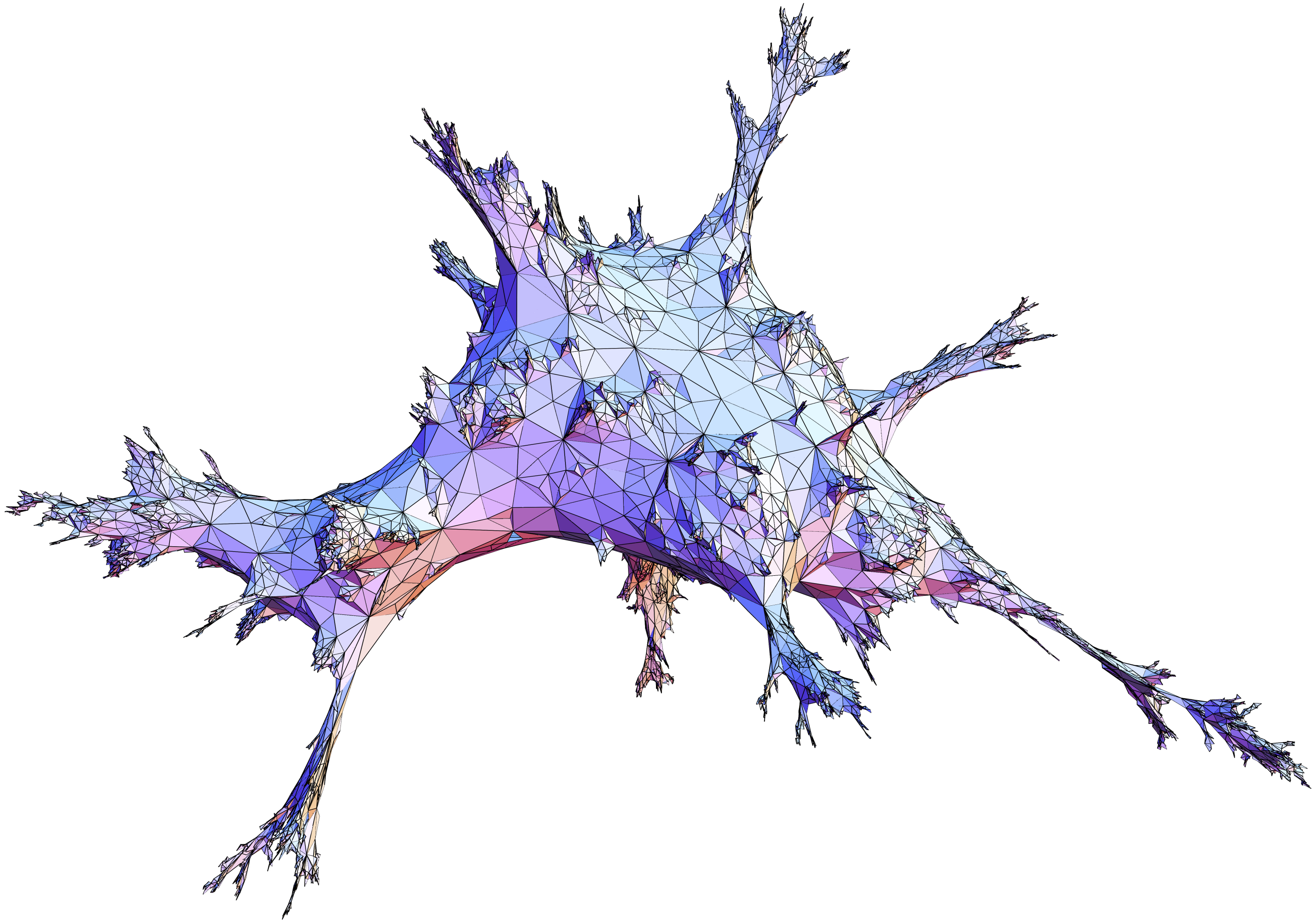
Then $M = (\mathcal{T}_e/\sim_{D^\star}, D^*)$ is the **Brownian map**.

The Brownian map



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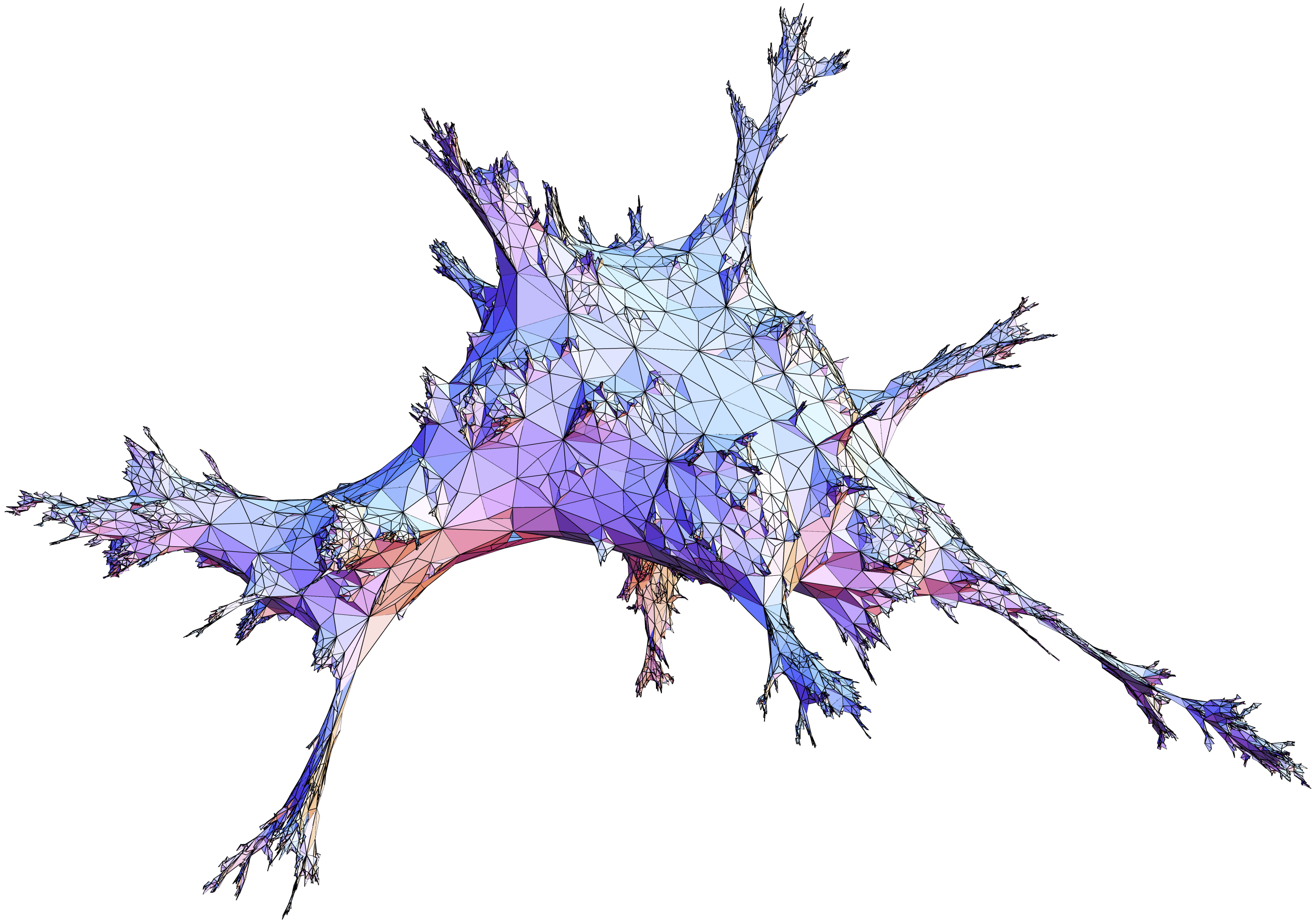
The Brownian map



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