Quantum invariants of knots and 3-manifolds

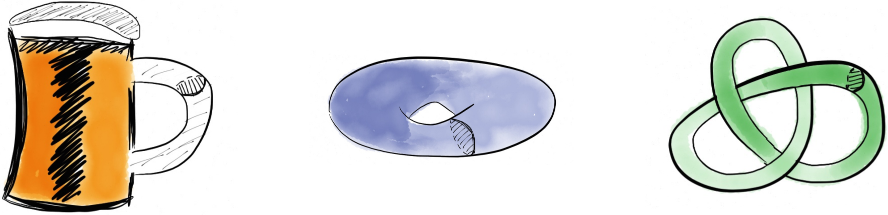
Clément Maria

The University of Queensland

June 2015

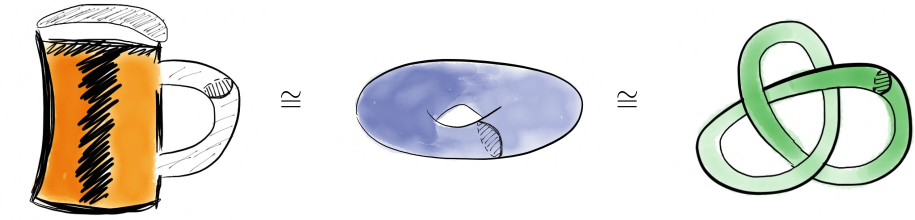
I. Topology of knots and manifolds

Topological equivalence(s)



- *Homeomorphism*: bijective continuous function with continuous inverse.
- *Isotopy*: continuous family of homeomorphism ("deformation").
- *Invariant*: property invariant under homeomorphism/isotopy.

Topological equivalence(s)



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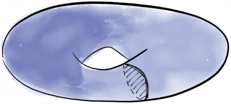

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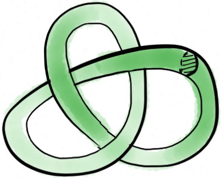

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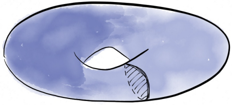

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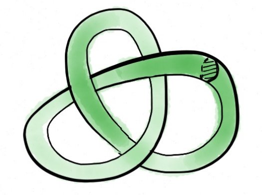

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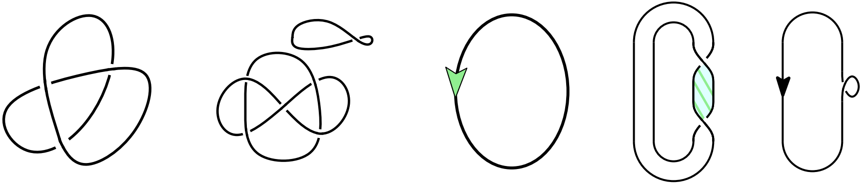
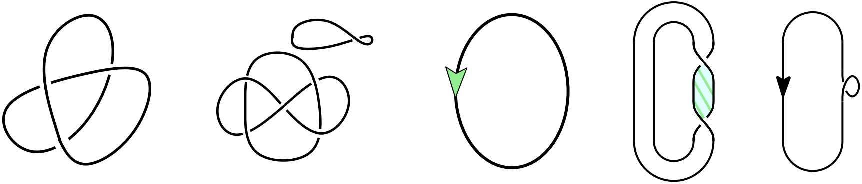
Knots, links and ribbons



- Knot: embedding of $S^1 \to \mathbb{R}^3$.
- Link: embedding of $S^1 \times \ldots \times S^1 \to \mathbb{R}^3$.
- Ribbon: knot/link with orientation and framing.

Knots, links and ribbons



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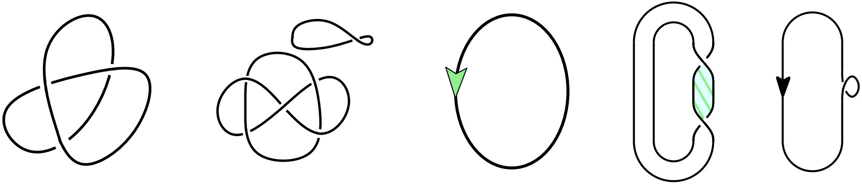

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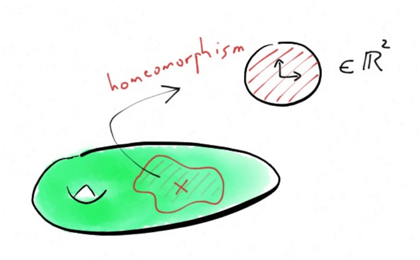

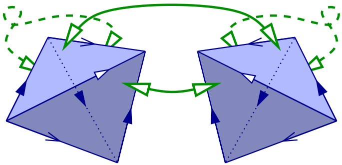

- *d*-manifold: every point is locally homeomorphic to $\mathbb{B}^d$.
- Generalized 3-triangulation: set of tetrahedra with triangle gluings.

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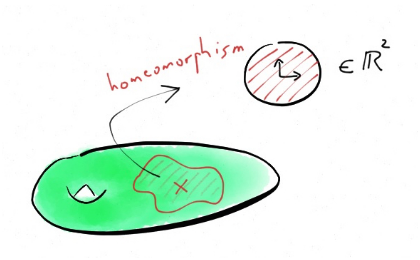

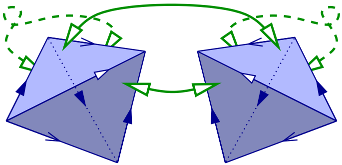

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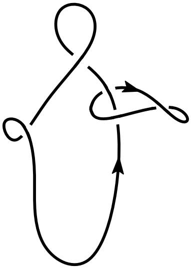
Quantum invariants of knots

Construction of the invariant



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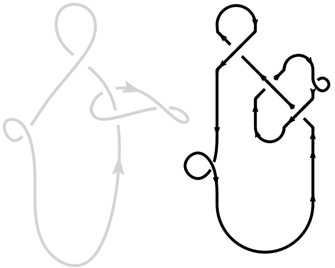

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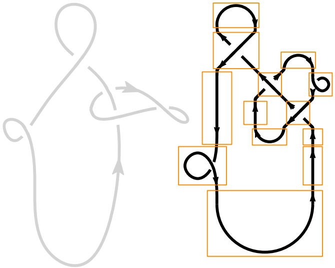

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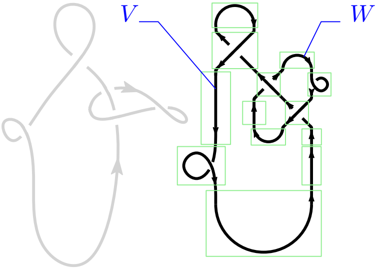

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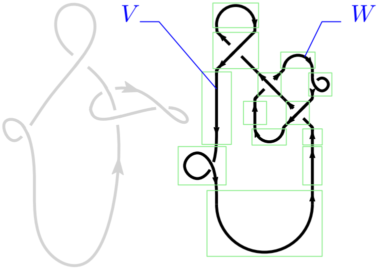
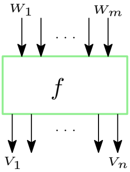
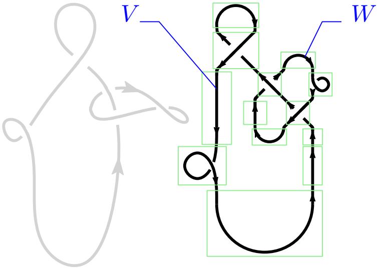
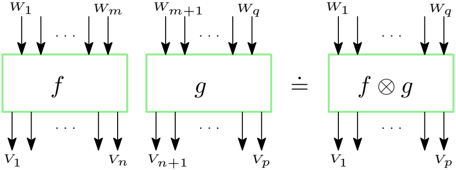




$$f\colon V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m$$

Construction of the invariant





$f \otimes g\colon V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots W_q$

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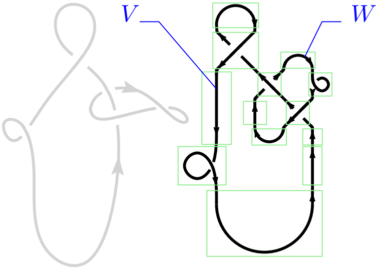

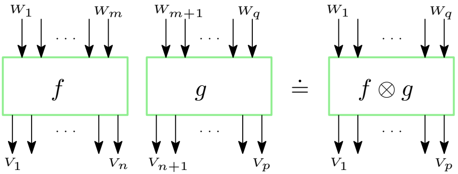

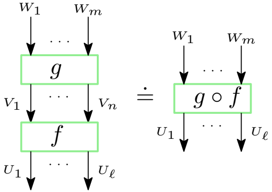

$f \otimes g \colon V_1 \otimes \ldots \otimes V_p \to W_1 \otimes \ldots W_q$

$g \circ f \colon U_1 \otimes \ldots \otimes U_\ell$

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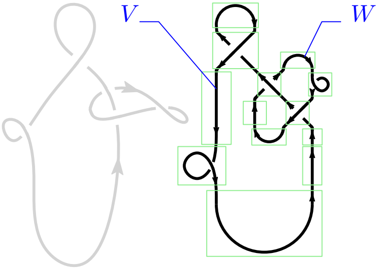
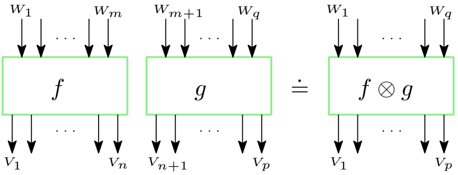


$V \; \downarrow \; \doteq \; \mathrm{id}_V \colon V \to V \qquad V \; \uparrow \; \doteq \; \mathrm{id}_{V^*}$



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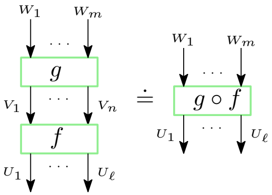

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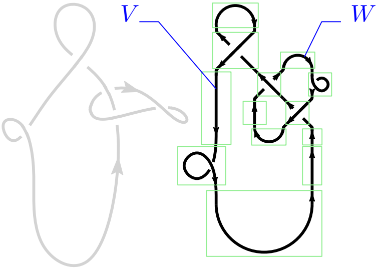

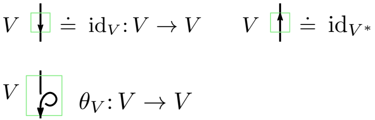

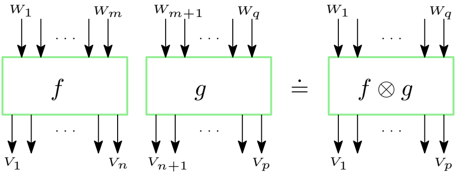

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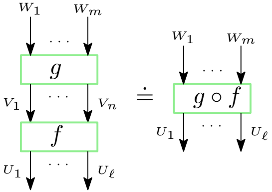

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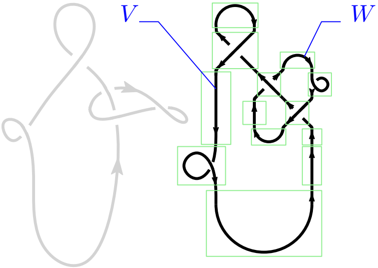

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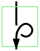
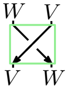
V  $\theta_V\colon V \to V$



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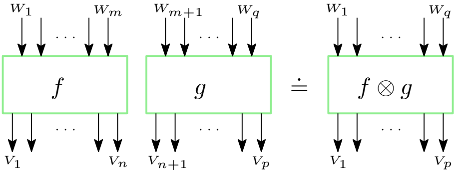

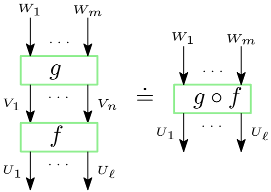

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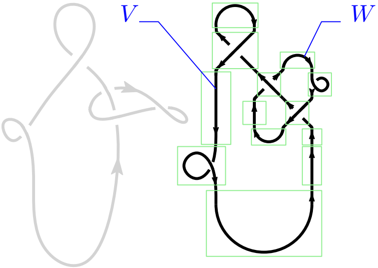

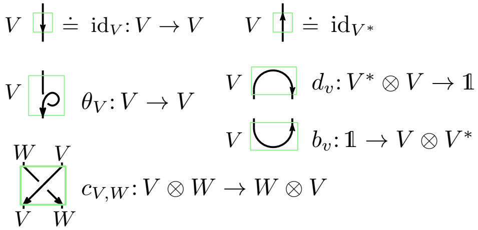

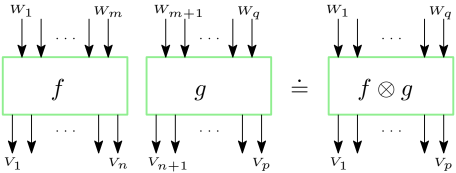

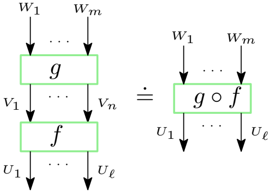

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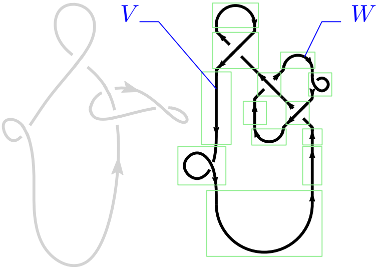

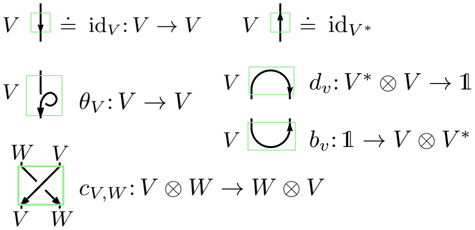

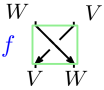

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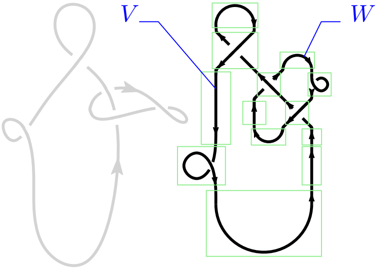

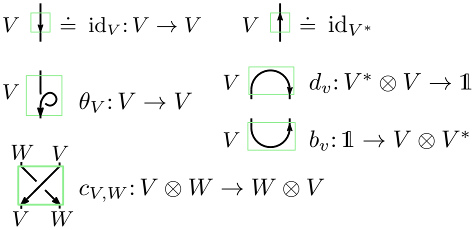

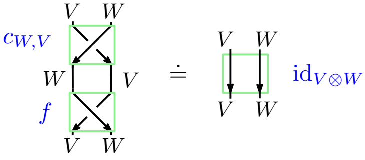

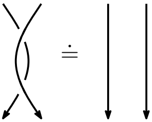

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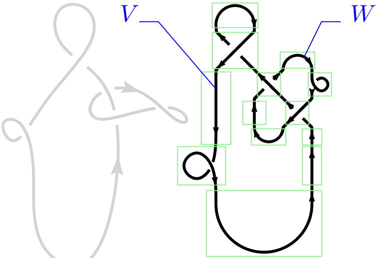

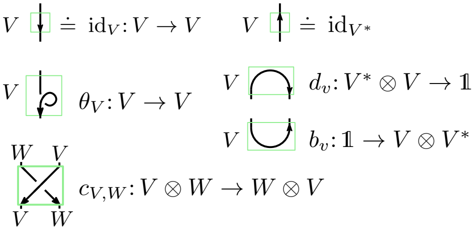

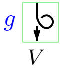

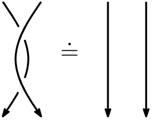

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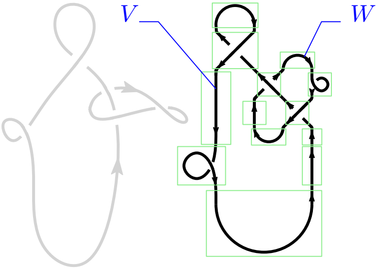

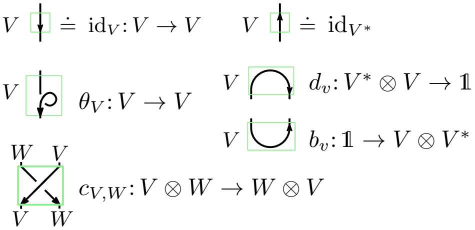

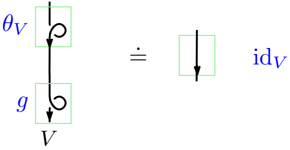

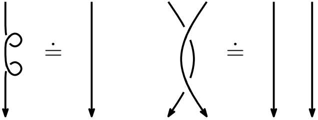

Ribbon category and ribbon diagrams

A *ribbon category* $\mathcal{V}$ is a category with:

- tensor product $\otimes \colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$,
- braiding $\{c_{V,W} \colon V \otimes W \to W \otimes V\}$,
- twist $\{\theta_V \colon V \to V\}$,
- duality $\{V^*, b_V \colon \mathbb{1} \to V \otimes V^*, d_V \colon V^* \otimes V \to \mathbb{1}\}$,

satisfying a set of natural axioms.

Theorem (Reshetikhin, Turaev)

A ribbon category associates to every $\mathcal{V}$-coloured ribbon diagram a morphism $\mathbb{1} \to \mathbb{1}$. It is an isotopy invariant.

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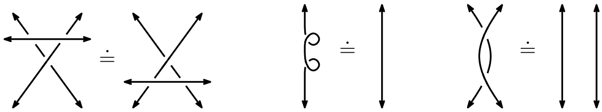
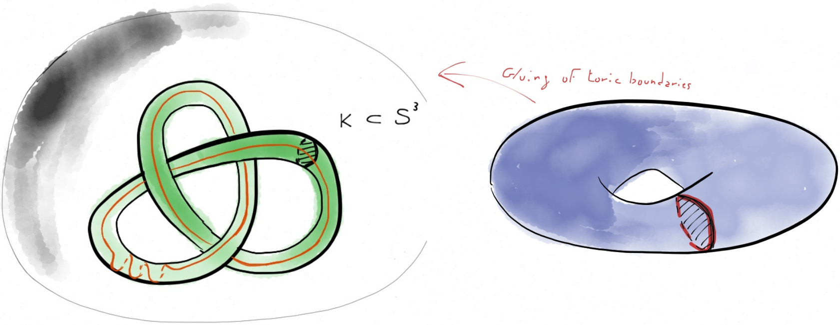
Proof: any isotopy of ribbon diagrams may be described by a sequence of *Reidemeister moves*.



Quantum invariants of 3-manifolds

Surgery presentation

Let $k \subseteq S^3$. A surgery on the 3-sphere along k consists in "drilling" k out of S^3 and glue back a solid torus along the toric boundary.

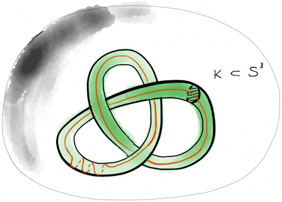


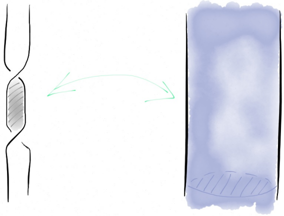
Theorem (Lickorish-Wallace)

Every 3-manifold may be obtained by surgery on S^3 along a link.

Invariant of 3-manifold

Let M be a 3-manifold, obtained by surgery on S^3 along a link k with m components $\{L_1, \ldots, L_m\}$.





Let $\mathcal{V}$ be a ribbon category [1]. For a colouring $\lambda\colon \{L_1, \ldots, L_m\} \to \mathcal{V}$, denote by $F(k, \lambda)$ the associated ribbon invariant.
Finally, sum over all colourings:

$$\tau(M, \mathcal{V}) = A_{\mathcal{V}} \sum_{\lambda\colon \{L_1,\ldots,L_m\} \to \mathcal{V}} D_\lambda \times F(k, \lambda)$$

[1] with an extra notion of "decomposability" of objects.

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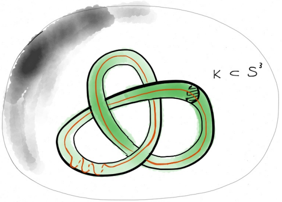
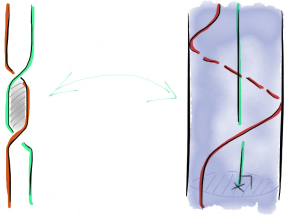
Let M be a 3-manifold, obtained by surgery on S^3 along a link k with m components $\{L_1, \ldots, L_m\}$.





Let $\mathcal{V}$ be a ribbon category [1]. For a colouring $\lambda\colon \{L_1, \ldots, L_m\} \to \mathcal{V}$, denote by $F(k, \lambda)$ the associated ribbon invariant.
Finally, sum over all colourings:

$$\tau(M, \mathcal{V}) = A_{\mathcal{V}} \sum_{\lambda\colon \{L_1,\ldots,L_m\} \to \mathcal{V}} D_\lambda \times F(k, \lambda)$$

[1] with an extra notion of "decomposability" of objects.

Invariant of 3-manifold

Theorem (Reshetikhin, Turaev)

For a manifold M obtained by surgery on S^3 along k, and a ribbon category $\mathcal{V}$,

$$\tau(M, \mathcal{V}) = A_{\mathcal{V}} \sum_{\lambda \colon \{L_1, \dots, L_m\} \to \mathcal{V}} D_\lambda \times F(k, \lambda)$$

is a 3-manifold invariant.

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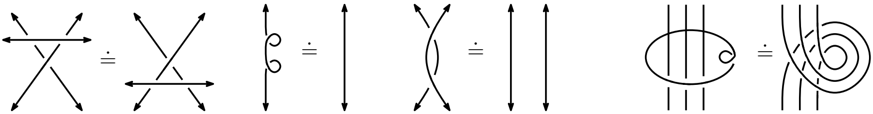
Proof: Two ribbons leading to the same manifold via surgery on S^3 are related by a sequence of Reidemeister moves and Kirby moves.

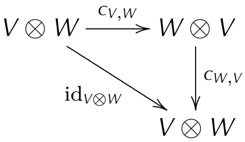


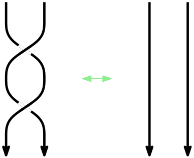
Why "quantum"?

It is easy to find algebraic objects (vector spaces, modules) with the structure of a ribbon category (usual tensor product, duality).

These simple examples however lead to trivial knots invariants.

Ex: vector spaces $c_{V,W}(v \otimes w) = w \otimes v$



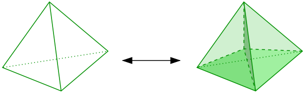


Quantum groups (in the representation theory of Lie algebras) lead to non-trivial ribbon categories. And powerful invairiants in $\mathbb{C}$.

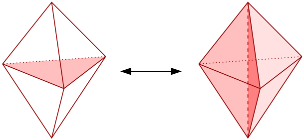
Algorithmic aspects of quantum invariants

Computation of the invariants

Pushing a bit more the construction, we get the *Turaev-Viro invariant* ($== |\tau|^2$) defined directly on the triangulation:



Pachner moves



Quantum groups lead to invariants parameterised by an integer $r \geq 3$.

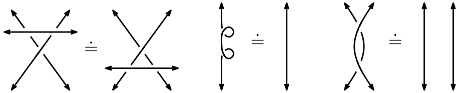
- $r = 3$, polynomial time algorithm (reduced to homology),
- $r = 4$, # P hard,
- fully parameterised algorithm in treewidth: $O((r+1)^{6k} \times \text{poly}(n))$

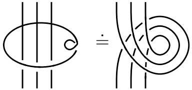
[Burton, M., Spreer '15]

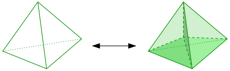
Conclusion

Take away

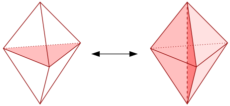
Turn a qualitative theory into a quantitative computation via Reidemeister moves, surgery, Kirby moves, Pachner moves, etc.







Pachner moves



Interesting complexity theory for the computation of quantum invariants.

Take away

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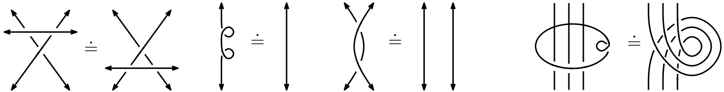
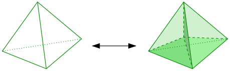
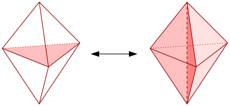




Pachner moves



Interesting complexity theory for the computation of quantum invariants.

Thank you!