Discrete Systolic Inequalities and Decompositions of Triangulated Surfaces

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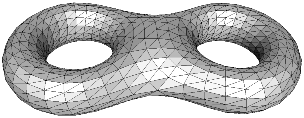
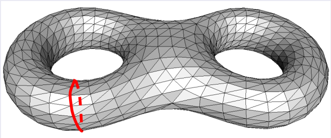


A primer on surfaces

We deal with ***connected***, ***compact*** and ***orientable*** surfaces of ***genus*** g without boundary.

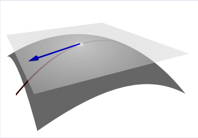
Discrete metric

Triangulation G.
Length of a curve $|\gamma|_G$:
Number of edges.



Riemannian metric

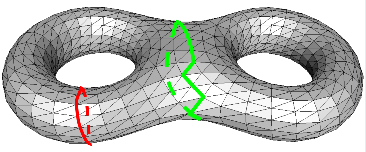
Scalar product m on the tangent space.
Riemannian length $|\gamma|_m$.



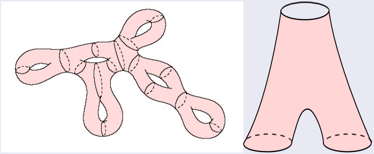
Systoles and surface decompositions

We study the length of topologically interesting curves and graphs, for discrete and continuous metrics.

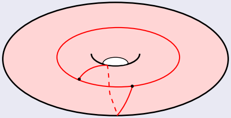
1.Non-contractible curves



2.Pants decompositions



3.Cut-graphs



Part 0:
Why should we care..

.. about graphs embedded on surfaces ?

- ***The easy answer***: because they are "natural". They occur in multiple settings: Graphics, computer-aided design, network design.
- ***The algorithmic answer***: because they are "general". Every graph is embeddable on some surface, therefore the genus of this surface is a natural parameter of a graph (similarly as tree-width, etc.).
- ***The hard answer***: because of Robertson-Seymour theory.

Theorem (Graph structure theorem, roughly)

Every minor-closed family of graphs can be obtained from graphs k-nearly embedded on a surface S, for some constant k.

Algorithms for surface-embedded graphs:

Cookie-cutter algorithm for surface-embedded graphs:

- Cut the surface into the plane.
- Solve the planar case.
- Recover the solution.

Examples: Graph isomorphisms, connectivity problems, matchings, expansion parameters, crossing numbers.

Algorithms for surface-embedded graphs:

Cookie-cutter algorithm for surface-embedded graphs:

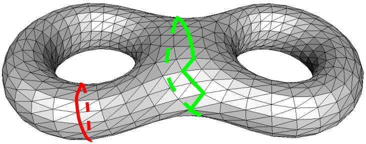
- Cut the surface into the plane.
  $\Rightarrow$ We need algorithms to do this cutting efficiently.
- Solve the planar case.
- Recover the solution.
  $\Rightarrow$ We need good bounds on the lengths of the cuttings.

Examples: Graph isomorphisms, connectivity problems, matchings, expansion parameters, crossing numbers.

Other motivations

- ***Topological graph theory:*** If the shortest non-contractible cycle is ***long***, the surface is ***planar-like***.
  $\Rightarrow$ Uniqueness of embeddings, colourability, spanning trees.
- ***Riemannian geometry:***
  René Thom: ***"Mais c'est fondamental !"***.
  Links with isoperimetry, topological dimension theory, number theory.
- More practical sides: ***texture mapping***, ***parameterization***, ***meshing*** ...

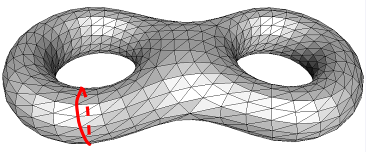
Part 1:
Cutting along curves



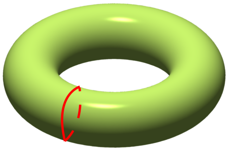
- Many results independently obtained by Ryan Kowalick in his PhD Thesis.

On shortest noncontractible curves

Discrete setting



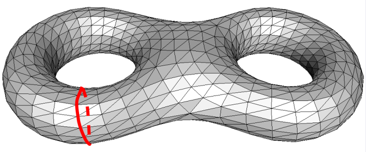
Continuous setting



What is the length of the red curve?

On shortest noncontractible curves

Discrete setting



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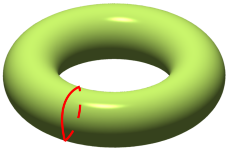
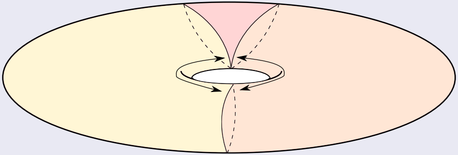


What is the length of the red curve?

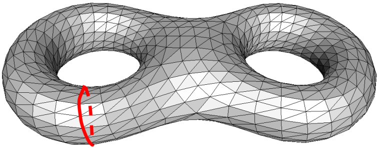
Intuition



It should have length $O(\sqrt{A})$ or $O(\sqrt{n})$, but what is the dependency on g ?

Discrete Setting: Topological graph theory

The ***edgewidth*** of a triangulated surface is the length of the shortest ***noncontractible*** cycle.



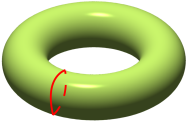
Theorem (Hutchinson '88)

The edgewidth of a triangulated surface with n triangles of genus g is $O(\sqrt{n/g}\log g)$.

- Hutchinson conjectured that the right bound is $\Theta(\sqrt{n/g})$.
- Disproved by Przytycka and Przytycki '90-97 who achieved $\Omega(\sqrt{n/g}\sqrt{\log g})$, and conjectured $\Theta(\sqrt{n/g}\log g)$.
- How about non-separating, or null-homologous non-contractible cycles ?

Continuous Setting: Systolic Geometry

The ***systole*** of a Riemannian surface is the length of the shortest ***noncontractible*** cycle.



Theorem (Gromov '83, Katz and Sabourau '04)

The systole of a Riemannian surface of genus g and area A is $O(\sqrt{A/g}\log g)$.

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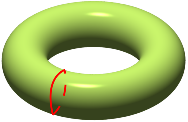

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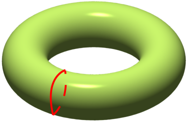

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- Buser and Sarnak '94 introduced ***arithmetic surfaces*** achieving the lower bound $\Omega(\sqrt{A/g}\log g)$.

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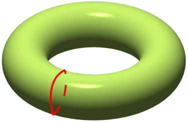

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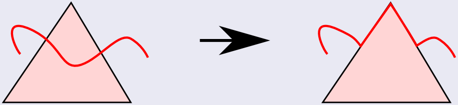
- Known variants for non-separating cycles and null-homologous non-contractible cycles [Sabourau '08].
- Buser and Sarnak '94 introduced ***arithmetic surfaces*** achieving the lower bound $\Omega(\sqrt{A/g}\log g)$.
- Larry Guth: "Arithmetic hyperbolic surfaces are remarkably hard to picture."

A two way street: From discrete to continuous

How to switch from a discrete to a continuous metric ?

Proof.

- Glue equilateral triangles of area 1 on the triangles .
- Smooth the metric.
- 
- In the worst case the lengths double. □

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, G) be a triangulated surface of genus g, with n triangles. There exists a Riemannian metric m on S with area n such that for every closed curve γ in (S, m) there exists a homotopic closed curve γ' on (S, G) with

$$|\gamma'|_G \leq (1+\delta)\sqrt[4]{3}\ |\gamma|_m \quad \text{for some arbitrarily small } \delta.$$

Corollary

Let (S, G) be a triangulated surface with genus g and n triangles.

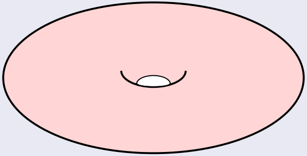
1. *Some non-contractible cycle has length $O(\sqrt{n/g}\log g)$.*
2. *Some non-separating cycle has length $O(\sqrt{n/g}\log g)$.*
3. *Some null-homologous non-contractible cycle has length $O(\sqrt{n/g}\log g)$.*

- (1) shows that Gromov $\Rightarrow$ Hutchinson and improves the best known constant.
- (2) and (3) are new.

A two way street: From continuous to discrete

How do we switch from a continuous to a discrete metric ?

Proof.



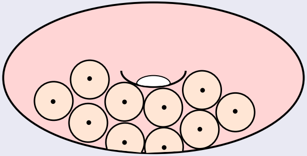
□

A two way street: From continuous to discrete

How do we switch from a continuous to a discrete metric ?

Proof.

Take a maximal set of balls of radius ε and perturb them a little.



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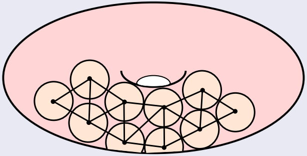

By [Dyer, Zhang and Möller '08], the Delaunay graph of the centers is a triangulation for ε small enough.

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A two way street: From continuous to discrete

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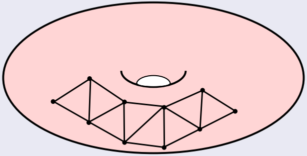
$\Rightarrow$ Triangulation T



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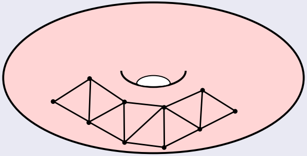

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$$|\gamma|_m \leq 4\varepsilon|\gamma|_G.$$

□

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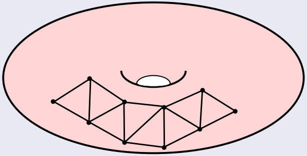

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$$|\gamma|_m \leq 4\varepsilon|\gamma|_G.$$

Each ball has radius $\pi\varepsilon^2 + o(\varepsilon^2)$, thus $\varepsilon = O(\sqrt{A/n})$.

□

Theorem and corollaries

Theorem (Colin de Verdière, Hubard, de Mesmay '14)

Let (S, m) be a Riemannian surface of genus g and area A. There exists a triangulated graph G embedded on S with n triangles, such that every closed curve γ in (S, G) satisfies

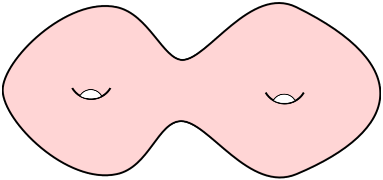
$|\gamma|_m \leq (1+\delta)\sqrt{\frac{32}{\pi}}\sqrt{A/n}\ |\gamma|_G$ *for some arbitrarily small δ.*

- This shows that Hutchinson $\Rightarrow$ Gromov.
- Proof of the conjecture of Przytycka and Przytycki:

Corollary

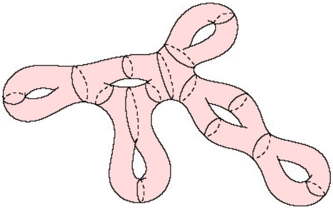
There exist arbitrarily large g and n such that the following holds: There exists a triangulated combinatorial surface of genus g, with n triangles, of edgewidth at least $\frac{1-\delta}{6}\sqrt{n/g}\log g$ *for arbitrarily small δ.*

Part 2: Pants decompositions



Pants decompositions

- A ***pants decomposition*** of a triangulated or Riemannian surface S is a family of cycles Γ such that cutting S along Γ gives pairs of pants, e.g., spheres with three holes.

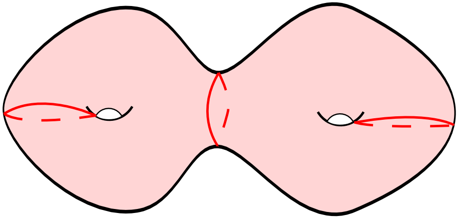


- A pants decomposition has $3g - 3$ curves.
- Complexity of computing a shortest pants decomposition on a triangulated surface: in NP, not known to be NP-hard.

Let us just use Hutchinson's bound

An algorithm to compute pants decompositions:

1. Pick a shortest non-contractible cycle.
2. Cut along it.
3. Glue a disk on the new boundaries.
4. Repeat $3g - 3$ times.



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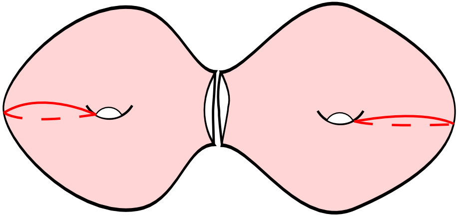

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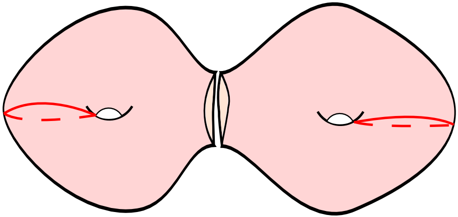

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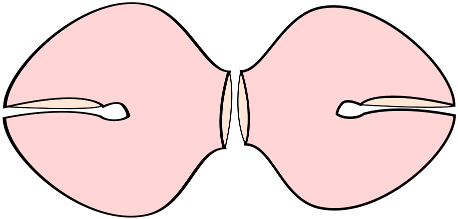


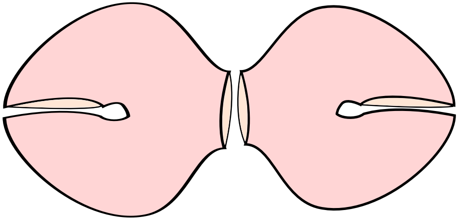
We obtain a pants decomposition of length

$$(3g - 3)O(\sqrt{n/g}\log g) = O(\sqrt{ng}\log g).$$

Let us just use Hutchinson's bound

An algorithm to compute pants decompositions:

1. Pick a shortest non-contractible cycle.
2. Cut along it.
3. Glue a disk on the new boundaries. ***This increases the area!***
4. Repeat $3g - 3$ times.



We obtain a pants decomposition of length

$$\sout{(3g-3)O(\sqrt{n/g}\log g) = O(\sqrt{ng}\log g).}\ \textit{Wrong!}$$

Doing the calculations correctly gives a subexponential bound.

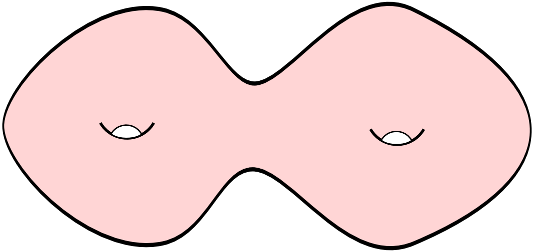
A correct algorithm

Denote by *PantsDec* the shortest pants decomposition of a triangulated surface.

- ***Best previous bound:*** $\ell(PantsDec) = O(gn)$.
  [Colin de Verdière and Lazarus '07]
- ***New result:*** $\ell(PantsDec) = O(g^{3/2}\sqrt{n})$.
  [Colin de Verdière, Hubard and de Mesmay '14]
- Moreover, the proof is algorithmic.

We "combinatorialize" a continuous construction of Buser.

First idea



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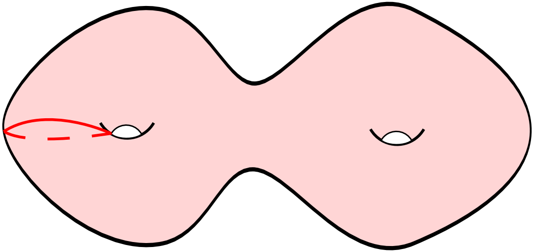

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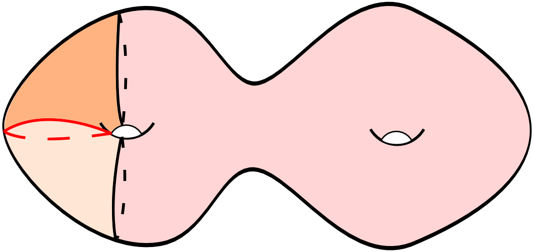
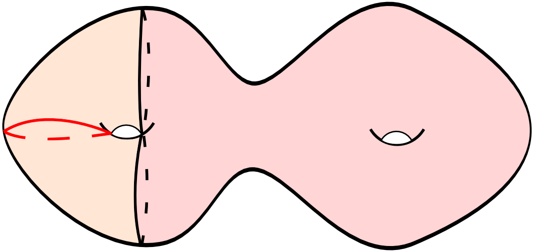


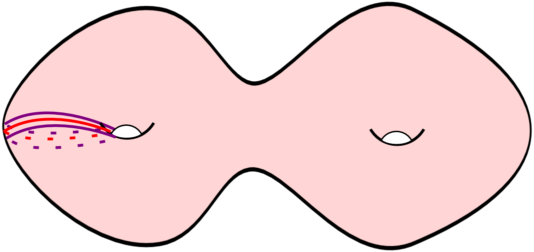
How to compute a short pants decomposition

First idea



If the torus is fat, this is too long.

~~First idea~~
Second idea



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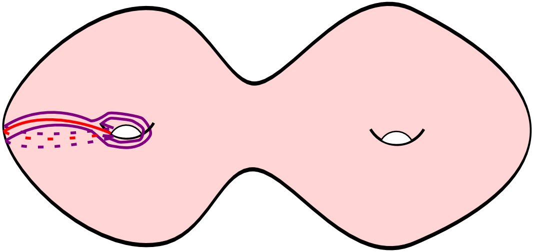
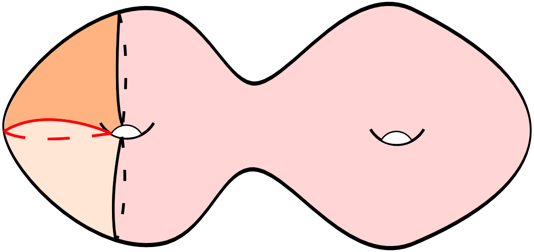


If the torus is thin, this is too long.

How to compute a short pants decomposition

~~*First idea*~~
~~*Second idea*~~
Both at the same time



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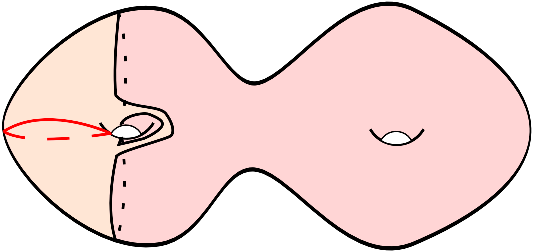
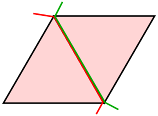


We take a trade-off between both approaches: As soon as the length of the curves with the first idea exceeds some bound, we switch to the second one.

- Several curves may run along the same edge: 

Pathologies

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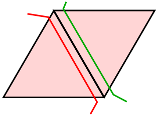

Random surfaces: Sample uniformly at random among the triangulated surfaces with n triangles.

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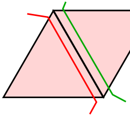
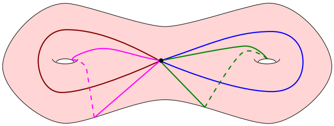


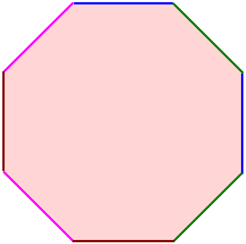
Random surfaces: Sample uniformly at random among the triangulated surfaces with n triangles.
These run-alongs happen a lot for random triangulated surfaces:

Theorem (Guth, Parlier and Young '11)

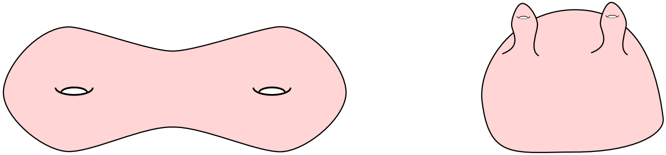
If (S, G) is a random triangulated surface with n triangles, and thus $O(n)$ edges, the length of the shortest pants decomposition of (S, G) is $\Omega(n^{7/6-\delta})$ w.h.p. *for arbitrarily small δ*

Part 3:
Cut-graphs with fixed combinatorial map





- What is the length of the shortest cut-graph with a fixed shape (combinatorial map) ?
- Useful to compute a homeomorphism between two surfaces.



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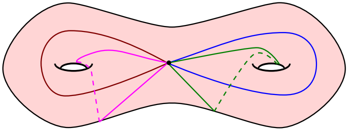

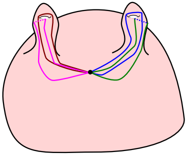

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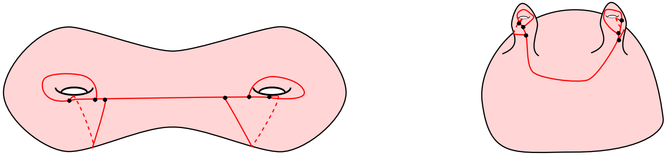

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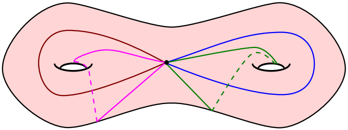

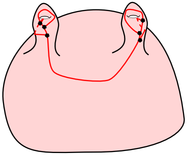

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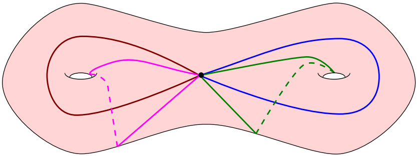

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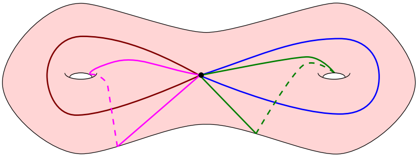

- Can one find a better map ?

Theorem (Colin de Verdière, Hubard, de Mesmay '13)

If (S, G) is a random triangulated surface with n triangles and genus g, for any combinatorial map M, the length of the shortest cut-graph with combinatorial map M is $\Omega(n^{7/6-\delta})$ w.h.p. for arbitrarily small δ.

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- How many surfaces of genus g with n triangles and cut-graph of length L? Roughly $L\,(L/g+1)^{12g-9}$.

- Restated in a dual setting: What is the minimal number of crossings between two cellularly embedded graphs G_1 and G_2 with specified combinatorial maps ?
- Related to questions of [Matoušek et al. '14] and [Geelen et al. '14].

Corollary

For a fixed G_1, for most choices of trivalent G_2 with n vertices, there are $\Omega(n^{7/6-\delta})$ crossings in any embedding of G_1 and G_2 for arbitrarily small δ.

Appendix: Discrete systolic inequalities in higher dimensions

- (M, T) : triangulated d-manifold, with $f_d(T)$ facets and $f_0(T)$ vertices.
- Supremum of $\frac{\mathrm{sys}^d}{f_d}$ or $\frac{\mathrm{sys}^d}{f_0}$?

Theorem (Gromov)

For every d, there is a constant C_d such that, for any Riemannian metric on any ***essential*** *compact d-manifold M without boundary, there exists a non-contractible closed curve of length at most $C_d \mathit{vol}(m)^{1/d}$.*

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  - Endow the metric of a regular simplex on every simplex.
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  - Push curves inductively to the 1-dimensional skeleton.

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Theorem (Gromov)

*For every d, there is a constant C_d such that, for any **piecewise** Riemannian metric on any **essential** compact d-manifold M without boundary, there exists a non-contractible closed curve of length at most $C_d vol(m)^{1/d}$.*

- We follow the same approach as for surfaces:
  - Endow the metric of a regular simplex on every simplex.
  - ~~Smooth the metric.~~ ***Non-smoothable triangulations*** [Kervaire '60]
  - Push curves inductively to the 1-dimensional skeleton.
- Corollary: $\frac{\mathrm{sys}^d}{f_d}$ is upper bounded by a constant for essential triangulated manifolds.

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Question: Are there manifolds M of dimension $d \geq 3$ for which there exists a constant c_M such that, for every triangulation (M, T), there is a non-contractible closed curve in the 1-skeleton of T of length at most $c_M f_0(T)^{1/d}$?

Recap

- The shortest non-contractible, non-separating and null-homologous non-contractible cycles on a triangulated surface have length $O(\sqrt{n/g}\log g)$ and this bound is tight.
- Our techniques generalize to higher dimensions.
- The shortest pants decomposition of a triangulated surface has length $O(g^{3/2}\sqrt{n})$ and we provide an algorithm to compute it.
- For random surfaces and any combinatorial map M, the length of the shortest cut-graph with combinatorial map M is $\Omega(n^{7/6-\delta})$.

Thank you ! Questions ?