Discrete planes: an arithmetic and dynamical approach

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http://www.liafa.univ-paris-diderot.fr/~berthe
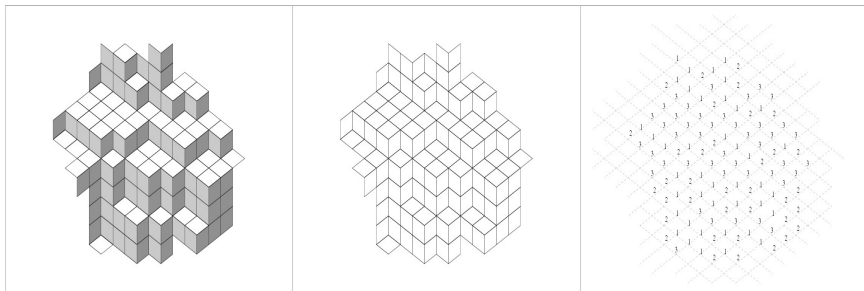
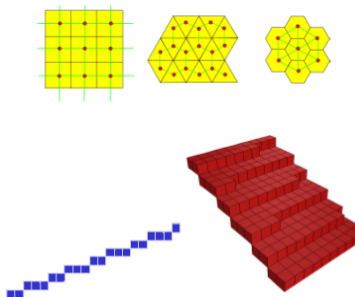*Séminaire de géométrie algorithmique et combinatoire*

From discrete geometry to word combinatorics...

...via tilings and quasicrystals

Discrete geometry Digital geometry

Analysis of geometric problems on objects defined on regular lattices


It requires

- a choice of a grid/lattice
- a topology
- basic primitives (lines, circles etc.)
- a dedicated algorithmics

Discrete planes

How to discretize a line in the space?

- There are the usual difficulties related to discrete geometry
- There are further difficulties due to the codimension > 1 for discrete lines

[D. Coeurjoly, Digital geometry in a Nutshell
http://liris.cnrs.fr/david.coeurjolly/doku/doku.php]
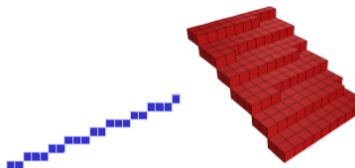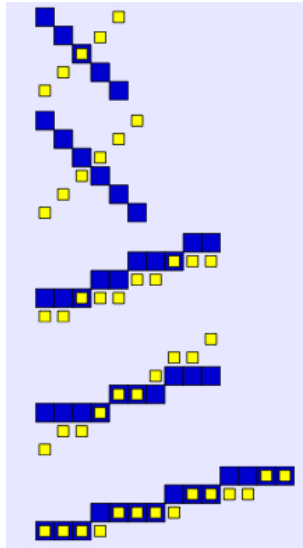
Euclid first axiom

Given two points A and B, there exists a unique line that contains them

This is no more true in the discrete case
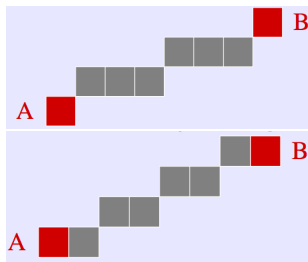
[D. Coeurjoly, Digital geometry in a Nutshell]

Intersections

[D. Coeurjoly, Digital geometry in a Nutshell]
[I. Sivignon, D. Coeurjoly, Introduction à la géométrie discrète]

Arithmetic discrete planes [Reveillès'91]

Let $\vec{v} \in \mathbb{R}^d$, $\mu, \omega \in \mathbb{R}$.

The arithmetic discrete plane $\mathfrak{P}(\vec{v}, \mu, \omega)$ is defined as

$$\mathfrak{P}(\vec{v}, \mu, \omega) = \{\vec{x} \in \mathbb{Z}^d \mid 0 \leq \langle \vec{x}, \vec{v} \rangle + \mu < \omega\}.$$

- μ is the translation parameter.
- ω is the width.
- If $\omega = \max_i\{|v_i|\} = ||\vec{v}||_\infty$, then $\mathfrak{P}(\vec{v}, \mu, \omega)$ is said naive.
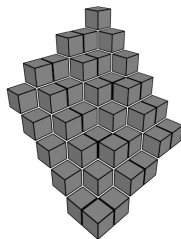- If $\omega = \sum_i |v_i| = ||\vec{v}||_1$, then $\mathfrak{P}(\vec{v}, \mu, \omega)$ is said standard.

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Reveillès'91, Françon, Andres, Debled-Renesson, Jacob-Dacol, Kiselman, Vittone, Chassery, Gérard, Buzer, Brimkov, Barneva, Rosenfeld, Klette...

Discrete lines and Sturmian words

Discrete lines

A discrete segment
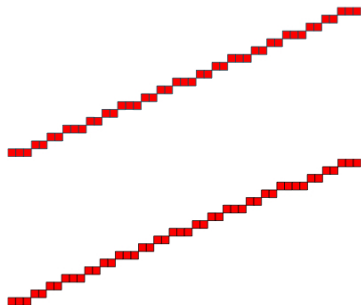
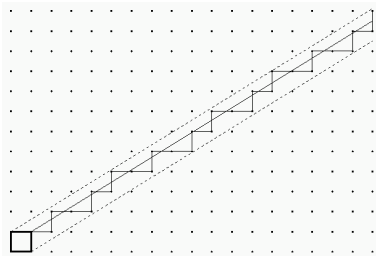and pixels that do not belong to a same discrete segment

[D. Coeurjoly, Digital geometry in a Nutshell]
[I. Sivignon, D. Coeurjoly, Introduction à la géométrie discrète]

Discrete lines and Sturmian words

One can code a discrete line (Freeman code) over the two-letter alphabet $\{0, 1\}$. One gets a Stumian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$

$$010010100100101010100100101$$

Discrete lines and Sturmian words

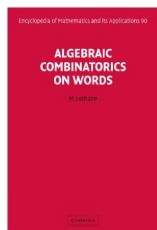One can code a discrete line (Freeman code) over the two-letter alphabet $\{0, 1\}$. One gets a Stumian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$

$$010010100100101001010010010101$$

[Lothaire, Algebraic combinatorics on words,
N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics
CANT Combinatorics, Automata and Number theory]

Discrete lines and Sturmian words
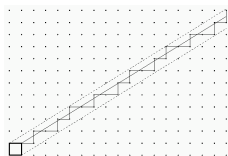
Let $R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, $x \mapsto x + \alpha \bmod 1$.

Sturmian words [Morse-Hedlund]

Let $(u_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ be a Sturmian word. There exist $\alpha \in (0,1)$, $\alpha \notin \mathbb{Q}$, $x \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N}, \ u_n = i \iff R_\alpha^n(x) = n\alpha + x \in I_i \ (\bmod\ 1),$$

with

$$I_0 = [0, 1-\alpha[, \ I_1 = [1-\alpha, 1[$$

or

$$I_0 =]0, 1-\alpha], \ I_1 =]1-\alpha, 1].$$

Factors

Theorem

The words 00 et 11 cannot be factors simultaneously of a Sturmian word

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The words 00 et 11 cannot be factors simultaneously of a Sturmian word

Preuve : One has

$$\forall i \in \mathbb{N}, \ u_n = i \iff n\alpha + x \in I_i \ (\text{mod } 1)$$

Hence

$$u_n u_{n+1} = 00$$

iff

$$\begin{cases} n\alpha + x \in [0, 1 - \alpha[\\ (n+1)\alpha + x \in [0, 1 - \alpha[\end{cases}$$

which requires $\alpha < 1/2$. One thus gets

$$u_n u_{n+1} = 00 \ \text{iff} \ n\alpha + x \in [0, 1 - 2\alpha[$$

From factors to intervals

$$R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \; x \mapsto x + \alpha \bmod 1$$

$1 - \alpha$

I_0

$1 - 2\alpha$

I_{00}

I_1 α
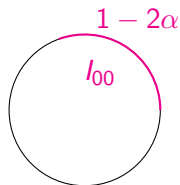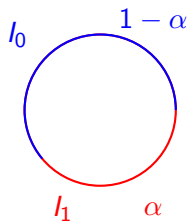
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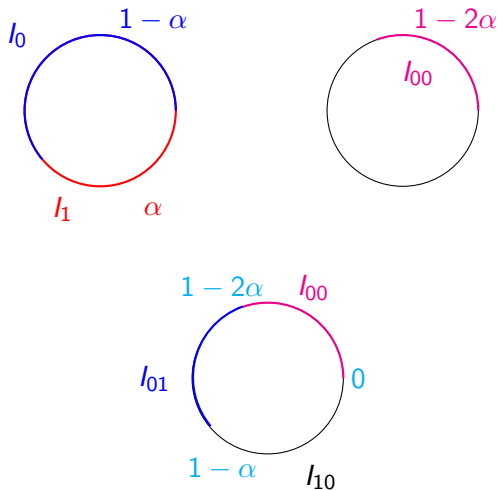
Property A Sturmian word has 3 factors of length 2

A key lemma

Let $I_0 = [0, 1 - \alpha[$, $I_1 = [1 - \alpha, 1[$. Let $R_\alpha \colon x \mapsto x + \alpha \bmod 1$.

Lemma The word $w = w_1 \cdots w_n$ over the alphabet $\{0, 1\}$ is a factor the Sturmian word u iff

$$I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} \neq \emptyset.$$

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Proof

$$\forall i \in \mathbb{N}, \ u_n = i \Longleftrightarrow n\alpha + x \in I_i \ (\bmod \ 1).$$

- One first notes that $u_k u_{k+1} \cdots u_{n+k-1} = w_1 \cdots w_n$ iff

$$\begin{cases} k\alpha + x \in I_{w_1} \\ (k+1)\alpha + x \in I_{w_2} \\ \dots \\ (k+n-1)\alpha + x \in I_{w_n} \end{cases}$$

- One then applies the density of $(k\alpha)_{\in \mathbb{N}}$ in $\mathbb{R}/\mathbb{Z}$.

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Application   One deduces combinatorial properties on the
- number of factors of given length/enumeration of local configurations
- densities of factors/statistical properties of local configurations
- powers of factors, repetitions, palindromes/symmetries

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Fact The sets $I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n}$ are intervals of $\mathbb{R}/\mathbb{Z}$.

The factors of u are in one-to-one correspondence with the $n + 1$ intervals of $\mathbb{T}$ whose end-points are given by

$$-k\alpha \bmod 1, \text{ for } 0 \leq k \leq n$$

Theorem [Coven-Hedlund]

A word $u \in \{0, 1\}^{\mathbb{N}}$ is Sturmian iff it admits eactly $n + 1$ factors of length n.

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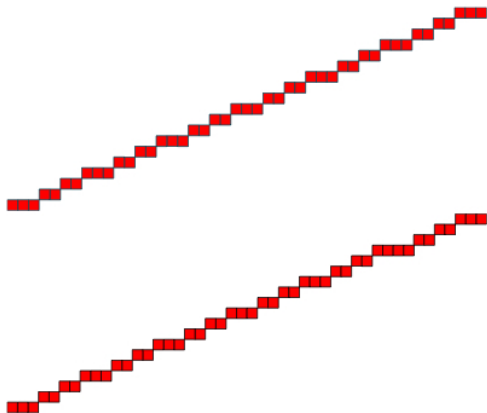
To summarize...

We have used

- A coding as an infinite binary word
- A dynamical system: the rotation of $\mathbb{R}/\mathbb{Z}$, $R_\alpha : x \mapsto x + \alpha$
- The key lemma: bijection between intervals and factors

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- The key lemma: bijection between intervals and factors

Discrete dynamical system

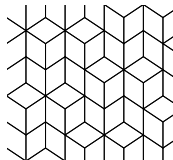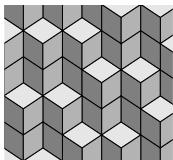A dynamical system (X, T) is defined as the action of a continuous and onto map T on a compact space X.

From a discrete plane to a tiling by projection....
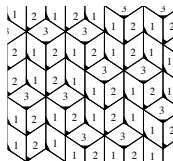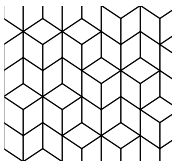
....and from a tiling by lozenges to a ternary coding
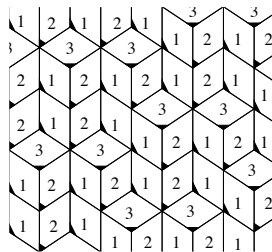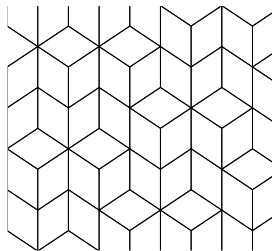
Two-dimensional word combinatorics

An arithmetic discrete plane can be coded as

1	2	1	2	1	2	3	1	2	1	2	3	1	3	2
3	1	3	1	2	1	2	3	1	2	1	2	1	2	2
2	1	2	3	1	2	1	2	3	1	3	1	2	1	2
1	2	1	2	3	1	3	1	2	1	2	3	1	2	1
3	1	2	1	2	1	2	3	1	2	1	2	3	1	2
2	3	1	3	1	2	1	2	3	1	2	1	2	1	2
1	2	1	2	3	1	2	1	2	3	1	3	1	2	1
3	1	2	1	2	3	1	3	1	2	1	2	3	1	2

Discrete planes and two-dimensional Sturmian words

Theorem [B.-Vuillon]

Let $(U_{m,n})_{(m,n)\in\mathbb{Z}^2} \in \{1,2,3\}^{\mathbb{Z}^2}$ be a 2d Sturmian word, that is, a coding of an arithmetic discrete plane. Then there exist $x \in \mathbb{R}$, and $\alpha, \beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly independent and $\alpha + \beta < 1$ such that

$$\forall (m,n) \in \mathbb{Z}^2,\ Um, n = i \iff R_\alpha^m R_\beta^n(x) = x + n\alpha + m\beta \in I_i \pmod 1,$$

with

$$I_1 = [0, \alpha[,\ I_2 = [\alpha, \alpha + \beta[,\ I_3 = [\alpha + \beta, 1[$$

or

$$I_1 =]0, \alpha],\ I_2 =]\alpha, \alpha + \beta],\ I_3 =]\alpha + \beta, 1].$$

Combinatorial properties of 2d Sturmian words

- They key lemma still holds: rectangular factors are in one-to-one correspondence with intervals of $\mathbb{R}/\mathbb{Z}$.

Theorem [B.-Vuillon]

There exist exactly $mn + m + n$ rectangular factors of size $m \times n$ in a 2d Sturmian word.

- Two discrete planes with the same normal vector have the same configurations.

- We also deduce information on the frequencies of configurations

[B.-Vuillon, Daurat-Tajine-Zouaoui]

Tilings of the line
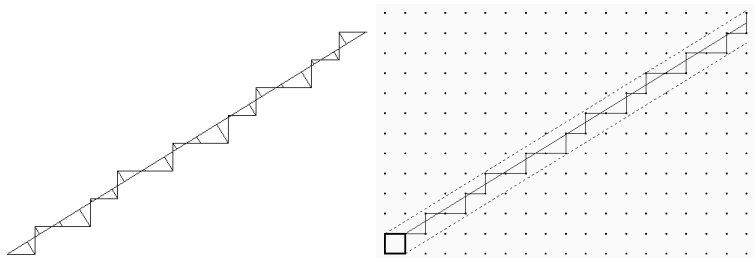
- By projecting the vertices of the discrete line, one gets a tiling of the line.
- This corresponds to a cut-and-project scheme in quasicrystallography.

Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure
that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order

(cf. Aperiodic tilings [Wang'61, Berger'66, Robinson'71,...)

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- Quasicrystals produce a discrete diffraction diagram (=order)
- Diffraction comes from regular spacing and local interactions of the point set Λ (consider the relative positions $\Lambda - \Lambda$)

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There are mainly two methods for producing quasicrystals
- Substitutions
- Cut and project schemes

[What is... a Quasicrystal? M. Senechal]

Cut and project schemes

Projection of a "plane" slicing through a higher dimensional lattice

- The order comes from the lattice structure
- The nonperiodicity comes from the irrationality of the normal vector of the "plane"
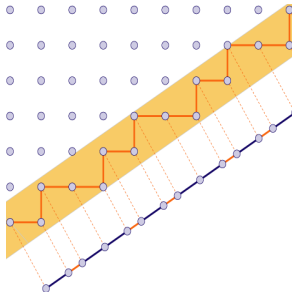
Recurrence and frequencies

Frequencies

The frequency f_w of a word w in $u = (u_n)_{n \in \mathbb{N}}$ is defined as the following limit, if it exists

$$f_w = \lim_{n \to \infty} \frac{|u_0 \cdots u_{N-1}|_w}{N}$$

where $|x|_j$ stands for the number of occurrences of w in x

By uniform distribution of $(k\alpha)_k$ modulo 1, the frequency of a factor w of a Sturmian word/discrete line is equal to the length of I_w

$$I_w = I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n}$$

Three-length theorem

Let $0 < \alpha < 1$ be an irrational number

Theorem The points $\{i\alpha\}$, for $0 \le i \le n$, partition the unit circle into $n+1$ intervals, the lengths of which take at most three values, one being the sum of the other two.

[Steinhaus, Sós, Świerczkowski, Surányi]

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[Steinhaus, Sós, Świerczkowski, Surányi]

Theorem The frequencies of factors of length n of a Sturmian word take at most 3 values

Three-gap theorem [Slater]

Let α be an irrational number in $]0, 1[$ and let I be an interval of $\mathbb{R}/\mathbb{Z}$.

The sequence $(n\alpha)_{n \in \mathbb{N}}$ mod 1 enters the interval I with bounded gaps, that is, there exists $N \in \mathbb{N}$ such that any sequence of N successive values of the sequence contains a value in I.

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Theorem The gaps between the successive integers j such that $\{\alpha j\} \in I$ take at most three values, one being the sum of the other two.

Recurrence function

The recurrence function R of a uniformly recurrent word u is defined by

$$R_u(n) = \min\{m \in \mathbb{N} \text{ such that } \forall B \in L_m, \forall A \in L_n, \text{ A is a factor of B}\}$$

where L_n denotes the set of factors of u of length n.

$R(n)$ is the size of the smallest window that contains all factors of length n whatever its position on the word.

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$R(n)$ is the size of the smallest window that contains all factors of length n whatever its position on the word.

Theorem Let u be a Sturmian word with angle α. Let $(q_k)_{k \in \mathbb{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α.

$$R(n) = n - 1 + q_k + q_{k-1}, \text{ where } q_{k-1} \le n < q_k.$$

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Recurrence function

- Let δ_n be the smallest length of the nonempty intervals $I_{w_1 \cdots w_n}$
- Let ℓ_n be the greatest gap between the successive integers k such that $\{k\alpha\} \in [0, \delta_n[$
- We have
$$R(n) = n - 1 + \ell_n.$$

Recurrence function

- Let δ_n be the smallest length of the nonempty intervals $I_{w_1 \cdots w_n}$

- Let ℓ_n be the greatest gap between the successive integers k such that $\{k\alpha\} \in [0, \delta_n[$

- We have
$$R(n) = n - 1 + \ell_n.$$

- Assume $q_{k-1} \leq n < q_k$. Then we have
$$\delta_n = \eta_{k-1} \text{ and } \ell_n = q_k + q_{k-1}.$$

- A Sturmian word is linearly recurrent iff its slope has bounded partial quotients
$$R(n) \leq Cn \quad \text{for all } n$$

Recurrence function of Sturmian words

- For any real α one has

$$\liminf_{n \to \infty} \frac{R_\alpha(n)}{n} \leq 3$$

- [Morse-Hedlund] For almost any real α, one has

$$\limsup \frac{R_\alpha(n)}{n \log n} = +\infty, \quad \text{and} \quad \limsup \frac{R_\alpha(n)}{n(\log n)^{1+\varepsilon}} = 0 \text{ for } \varepsilon > 0.$$

Recurrence and frequencies

Consider an infinite word such that all its factors admit frequencies

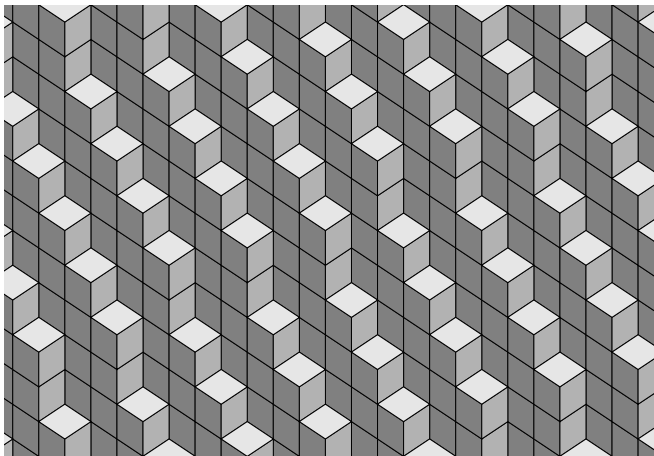- Let e_n be the smallest frequency of factors of length n

- Theorem Linear recurrence is equivalent to
  $$\exists C > 0, \ ne_n > C \text{ for all } n.$$

- One has for all n
  $$e_n \geq \frac{1}{R(n)}$$

	1	2	3	1	3	1	2	1	2	3	1	2	1	
1	2	1	2	1	2	3	1	2	1	2	3	1	3	
3	1	3	1	2	1	2	3	1	2	1	2	1	2	
2	1	2	3	1	2	1	2	3	1	3	1	2	1	
1	2	1	2	3	1	3	1	2	1	2	3	1	2	
3	1	2	1	2	1	2	3	1	2	1	2	3	1	
2	3	1	3	1	2	1	2	3	1	2	1	2	1	
1	2	1	2	3	1	2	1	2	3	1	3	1	2	
3	1	2	1	2	3	1	3	1	2	1	2	3	1	

Factors

- The block $W = [w_{i,j}]$, defined on $\{1, 2, 3\}$ and of size (m, n), is a factor of U if and only if

$$I_W := \bigcap_{1 \leq i \leq m, 1 \leq j \leq n} R_\alpha^{-i+1} R_\beta^{-j+1} I_{w_{i,j}} \neq \emptyset.$$

- The sets I_W are connected
- The frequency $f(W)$ of a factor W of the word U is defined as the limit, if it exists, of the number of occurrences of this block in the "central" square factor

$$\begin{matrix} U_{-n,n} & \dots & U_{n,n} \\ \vdots & & \vdots \\ U_{-n,-n} & \dots & U_{n,-n}, \end{matrix}$$

of the word divided by $(2n+1)^2$.

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- The frequency $f(W)$ of a factor W of the word U is defined as the limit, if it exists, of the number of occurrences of this block in the "central" square factor

$$\begin{matrix} U_{-n,n} & \dots & U_{n,n} \\ \vdots & & \vdots \\ U_{-n,-n} & \dots & U_{n,-n}, \end{matrix}$$

of the word divided by $(2n+1)^2$.

- The frequency of every factor W of U exists and is equal to the length of $I(W)$.

Number of frequencies

Theorem [Geelen and Simpson] The set of points

$$\{i\alpha + j\beta + \rho, \ 0 \le i \le m - 1, \ 0 \le j \le n - 1\}$$

partitions the unit circle into intervals having at most $\min\{m, n\} + 3$ lengths.

Frequencies The frequencies of rectangular factors of size (m, n) of a $2d$ Sturmian word take at most $\min\{m, n\} + 5$ values.

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Theorem [Chevallier] Let $d \ge 3$. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{T}^1$ and $2 \le n_1 \le \ldots \le n_d$ integers. The set

$$\left\{ \sum_{i=1}^{d} k_i \alpha_i,\ 0 \le k_i < n_i,\ i = 1, \ldots, d \right\}$$

divides $\mathbb{T}^1$ into intervals whose lengths take at most

$$\prod_{i=1}^{d-1} n_i + 3 \prod_{i=1}^{d-2} n_i + 1 \text{ values.}$$

Are there finitely many frequencies?

Are there finitely many lengths for the intervals obtained by taking points on $\mathbb{R}/\mathbb{Z}$

$$i\alpha + j\beta \quad 0 \leq i \leq m, \ 0 \leq j \leq n$$

- Is there a finite uniform upper bound on the number of distinct frequencies for rectangular factors of size (m, n) for some parameters (α, β)?
- Can one characterize those parameters?
- There are finitely many lengths for badly approximable numbers [Boshernitzan]

$$|(r, s)|^2 ||r\alpha + s\beta|| \geq C \text{ for all } (r, s) \neq 0$$

- What is the generic behavior?
- Same questions for squares?

Repetitivity

Fact Arithmetic discrete planes are repetitive (factors occur with bounded gaps)

Recurrence function Let N be the smallest integer N such that every square factor of radius N contains all square factors of size n. We set $R(n) := N$.

**Linear recurrence** There exists C such that $R(n) \leq Cn$ for all n.

Discrete planes [A. Haynes, H. Koivusalo, J. Walton] Linearly recurrent discrete planes are the planes that have a badly approximable normal vector

$$|(r,s)|^2 \|r\alpha + s\beta\| \geq C \text{ for all } (r,s) \neq 0$$

Discrete lines One has linear recurrence iff and the slope of the line has bounded partial quotients in its continued fraction expansion.

Substitutions

- Substitutions on words and symbolic dynamical systems
- Substitutions on tiles : inflation/subdivision rules, tilings and point sets

- Tilings Encyclopedia http://tilings.math.uni-bielefeld.de/
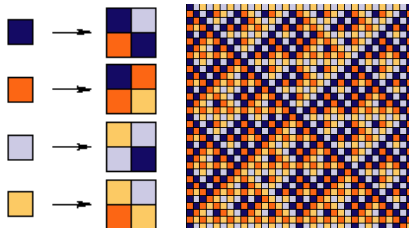
[E. Harriss, D. Frettlöh]

Back to tilings and long-range aperiodic order

Discrete planes with irrational normal vector are

- repetitive (uniform recurrence)
- aperiodic

The corresponding tilings are obtained by a cut and project scheme and yield quasicrystals

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Assume we have a "substitutive" arithmetic discrete plane

Multidimensional substitutive tilings ⤳ Local/matching rules
[S. Mozes, C. Goodman-Strauss]

One can recognize a given "substitutive" arithmetic discrete plane
by local inspection [N. Bedaride-Th. Fernique]