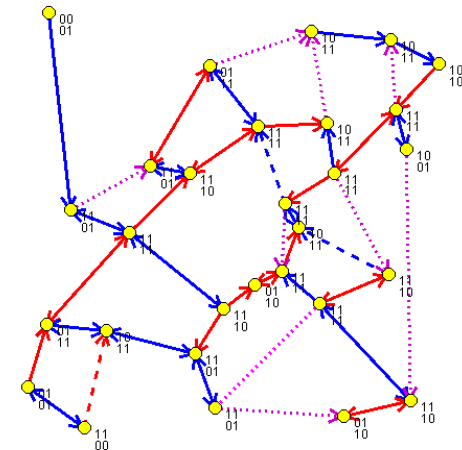
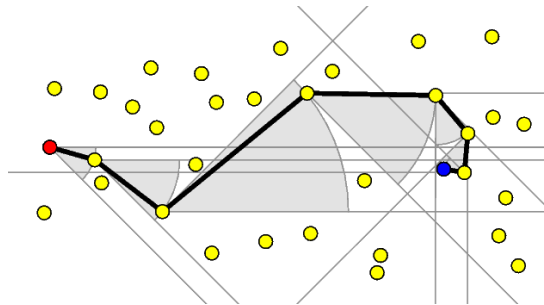
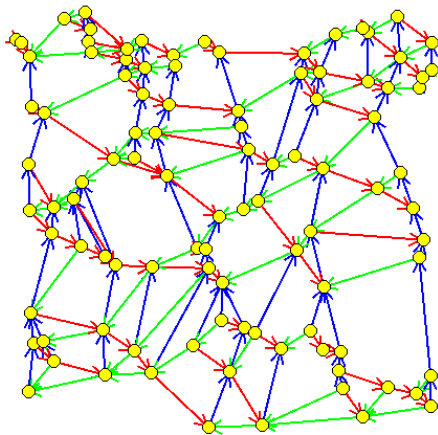
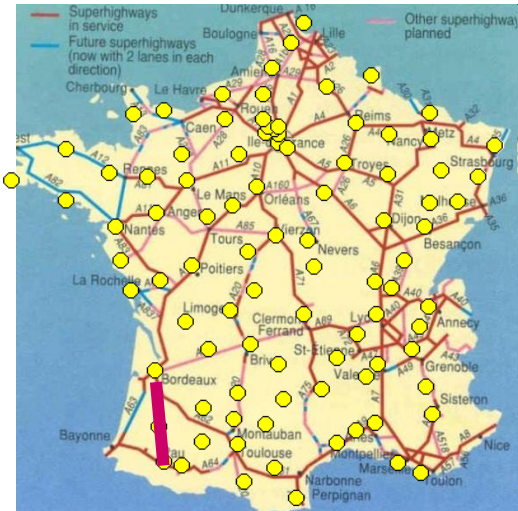


# *Etirement de quelques graphes géométriques*



*Nicolas Bonichon*

# Qu'est-ce qu'un bon réseau ?



**Etirement**  $:= \max_{a,b} \frac{\text{distance dans le réseau } a \rightarrow b}{\text{distance à vol d'oiseau } a \rightarrow b}$

**Nombre d'arêtes** ou somme des longueurs des arêtes

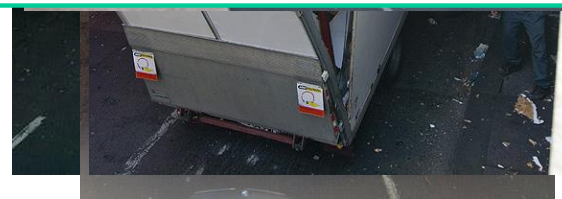
**Degré maximal** : taille du plus grand rond-point

**Planaire** : ni pont ni tunnel



[www.2m40.com](http://www.2m40.com)

50 accidents depuis mai 2009



# Modélisation du problème (but du jeu)

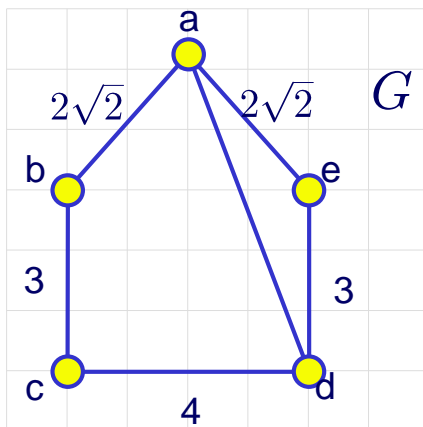
Etant donné un ensemble  $V$  de points dans le plan :

Construire un réseau (Graphe)  $G$  constitué de segments (arêtes) reliant ces points, tel que

- 1) l'étirement soit faible
- 2) les segments ne se croisent pas.

$$\text{Détour}(s,t) : \frac{d_G(s,t)}{||st||}$$

$$\text{Etirement}(G) : \max_{s \neq t} \text{Détour}(s,t)$$



Exemple :

$$d_G(b,d) = 3 + 4, ||bd|| = 5$$

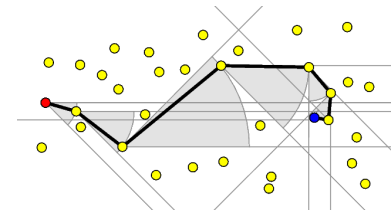
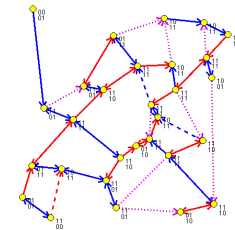
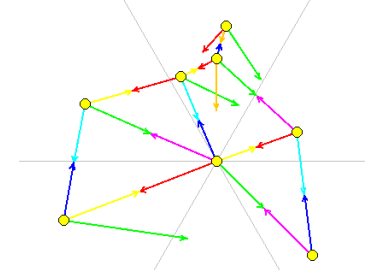
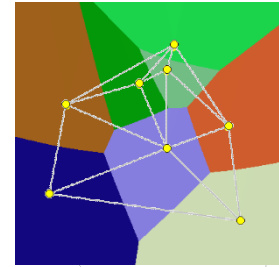
$$d_G(b,e) = 4\sqrt{2}, ||be|| = 4.$$

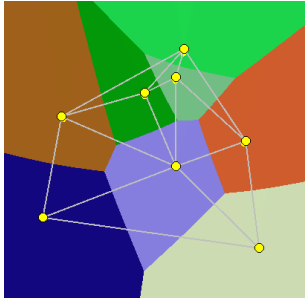
$G$  a un étirement de  $\sqrt{2}$ .

Un graphe est un  $t$ -spanner si son étirement est au plus  $t$ .

# Spanners

- Triangulations de Delaunay
- Théta-Graphs
- Spanners de Degré Borné
- Routage dans les Théta-graphes





# Delaunay Triangulation

[Boris Delaunay '34]

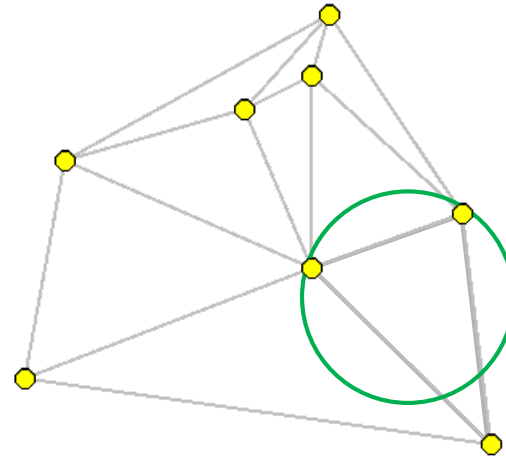
**Def:** There is an edge  $(i, j)$  iff there is an empty circle supporting  $i$  and  $j$ .

$\Leftrightarrow$ : there is a face  $i, j, k$  iff there is an empty circle supporting  $i, j, k$ .

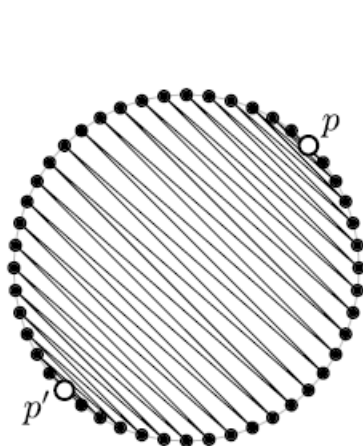
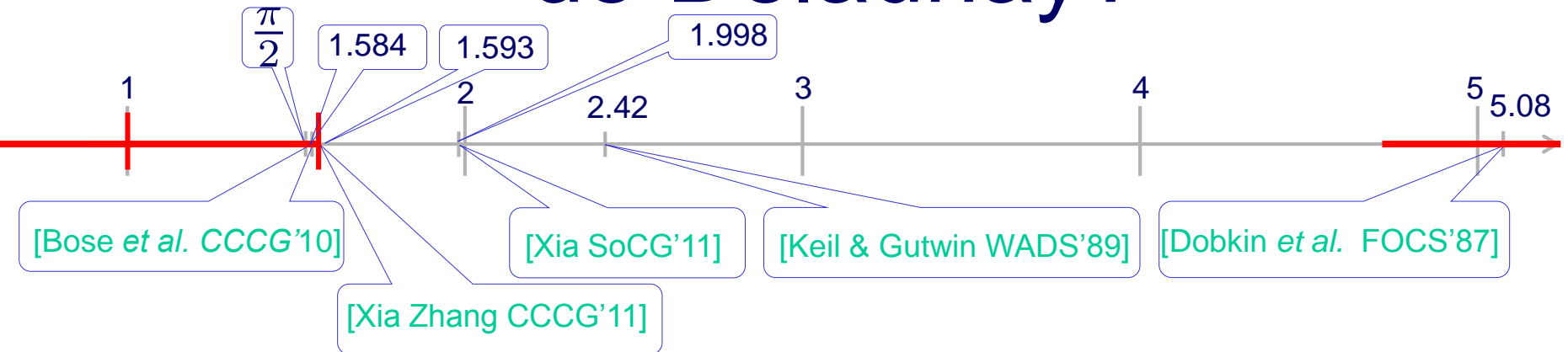
**General Position:**

No 3 points collinear

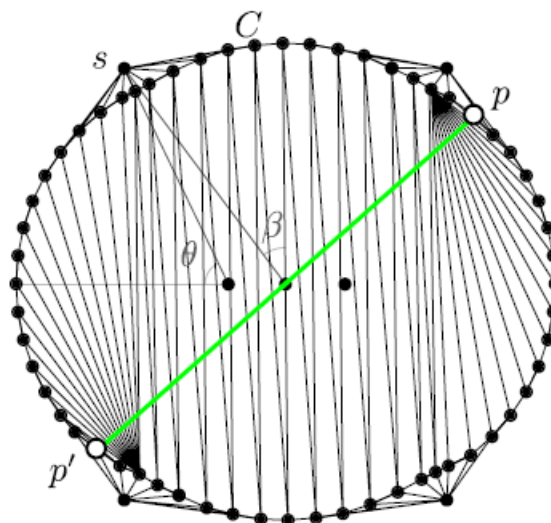
No 4 points co-circular.



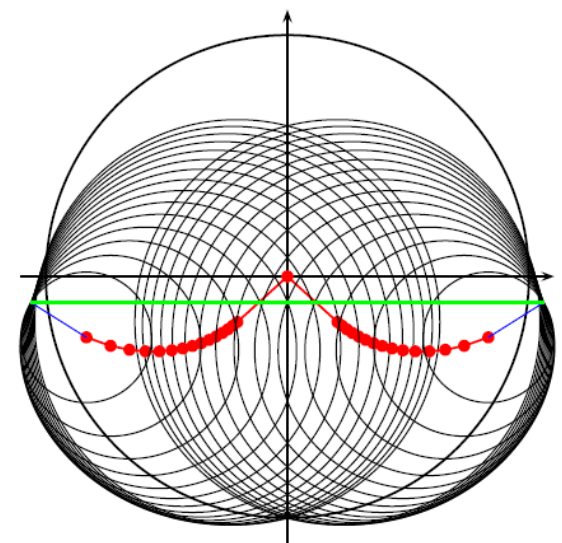
# Étirement des triangulations de Delaunay?



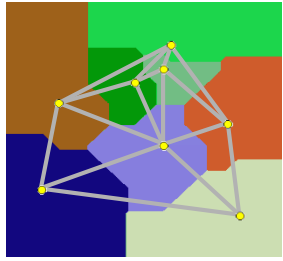
$$\pi/2 \approx 1.57079$$



$$1.584$$



$$1.593$$



# $L_\infty$ -Delaunay Triangulation

## [Chew SoCG'86]

$$d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)$$

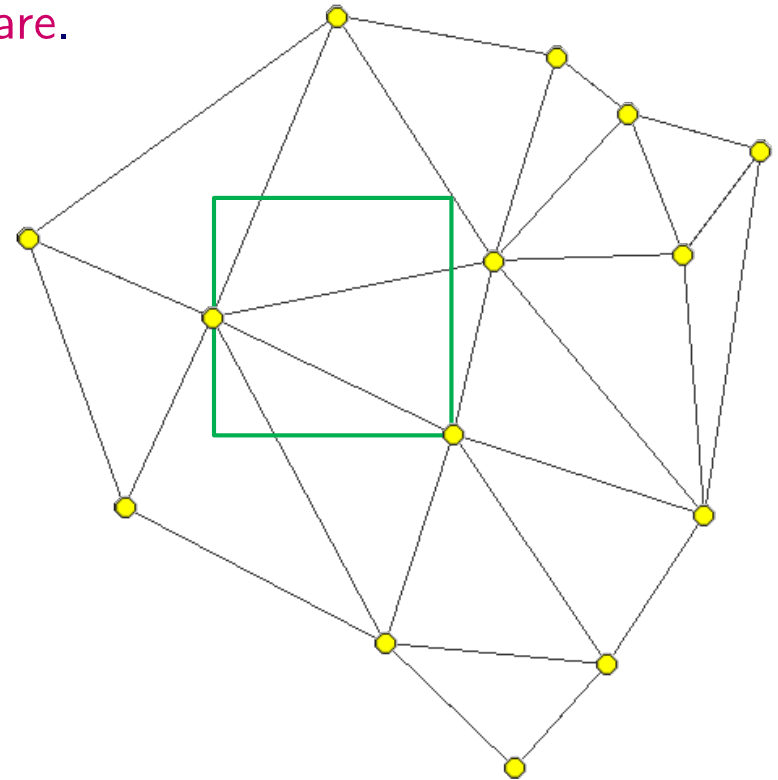
**Def:** There is an edge  $(i, j)$  iif there is an empty **square** supporting  $i$  and  $j$ .

$\Leftrightarrow$ : each face is supported by an empty **square**.

**General Position:**

No 4 points co-”circular”.

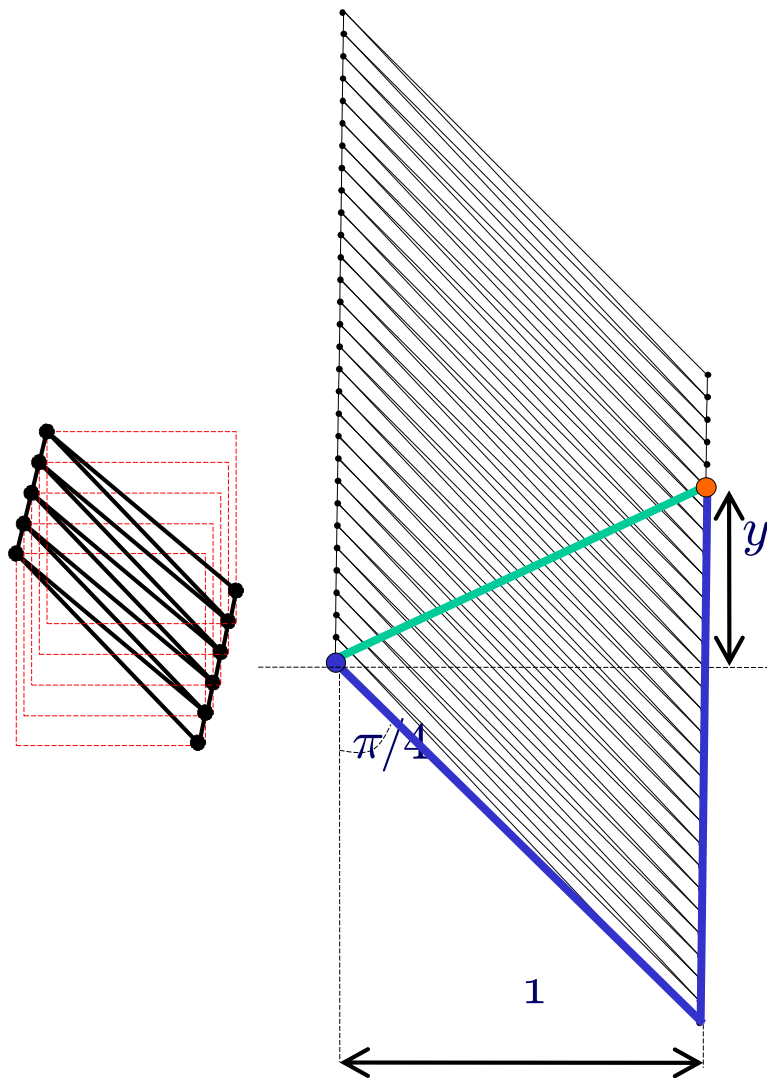
Points have  $\neq$  abscissas and ordinates.



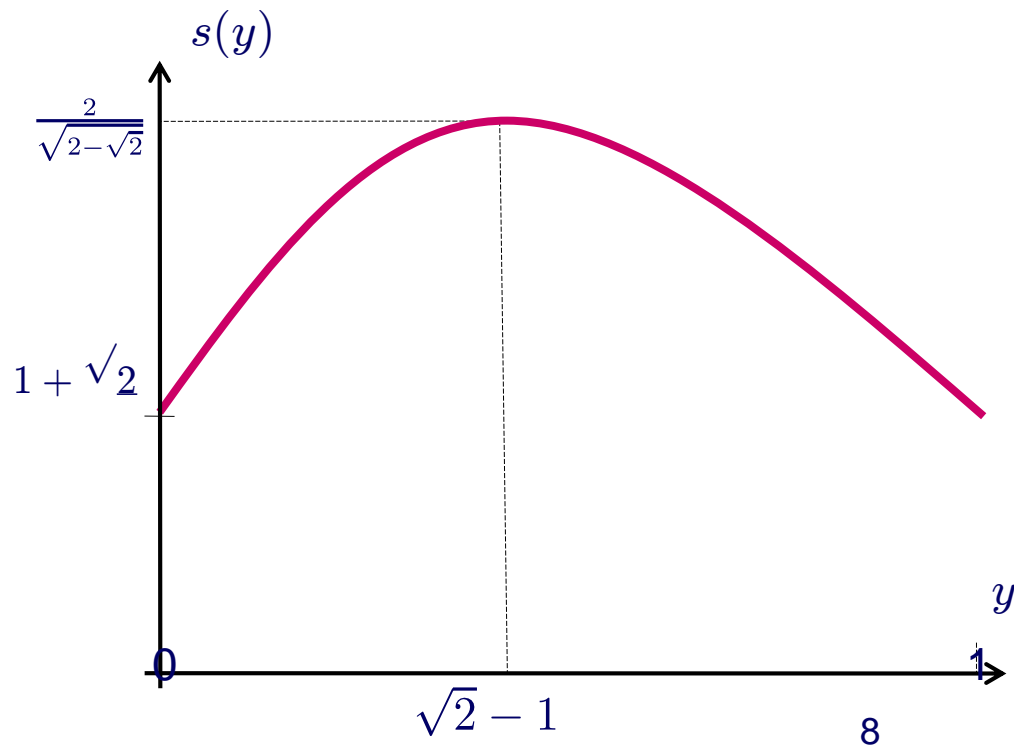
**Thm** [B. Gavoille Hanusse Perković 10]: Delaunay triangulation is 2.61-spanner.



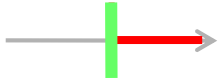
# Stretch $\geq 2.61$



$$s(y) = \frac{\sqrt{2} + 1 + y}{\sqrt{1 + y^2}}$$



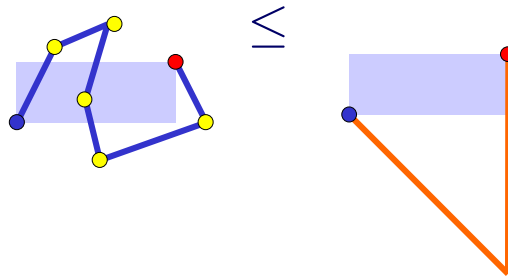




# Stretch $\leq 2.61$

Let us consider the  $L_\infty$ -Delaunay Triangulation  $T$  on a set of points  $P$  in the plane, and let  $a$  and  $b$  be two points of  $P$ . If  $x = d_\infty(a, b) = \max\{d_x(a, b), d_y(a, b)\}$  and  $y = \min\{d_x(a, b), d_y(a, b)\}$  then

$$d_T(a, b) \leq (1 + \sqrt{2})x + y.$$

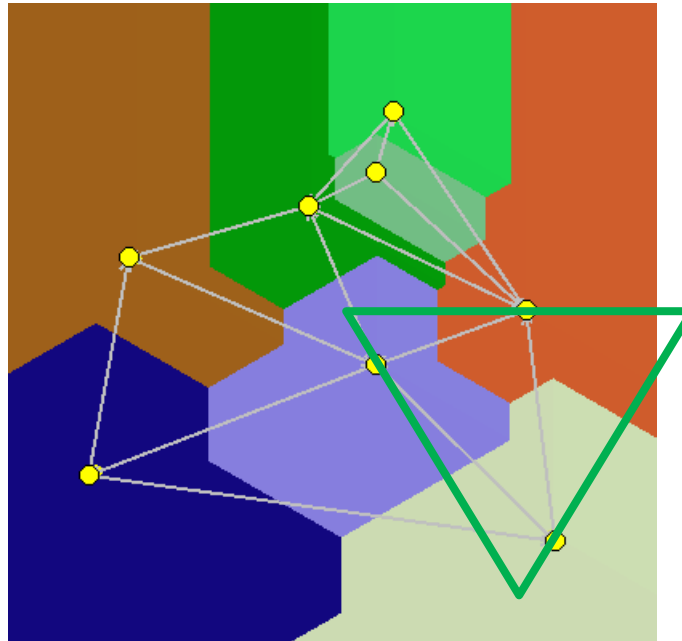


Proof: by induction on  $d_\infty(a, b) \dots$

# Triangulation de TD-Delaunay







[Chew 89]

TD : Triangular Distance



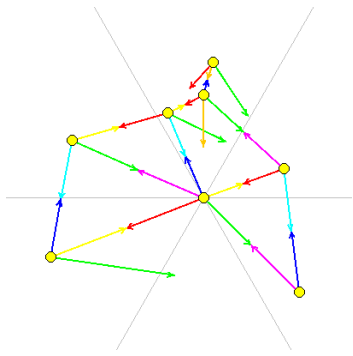
**Thm** [Chew 89] : Les triangulations de TD-Delaunay sont des 2-spanners planaires.

# Etirement *pgone*-Delaunay

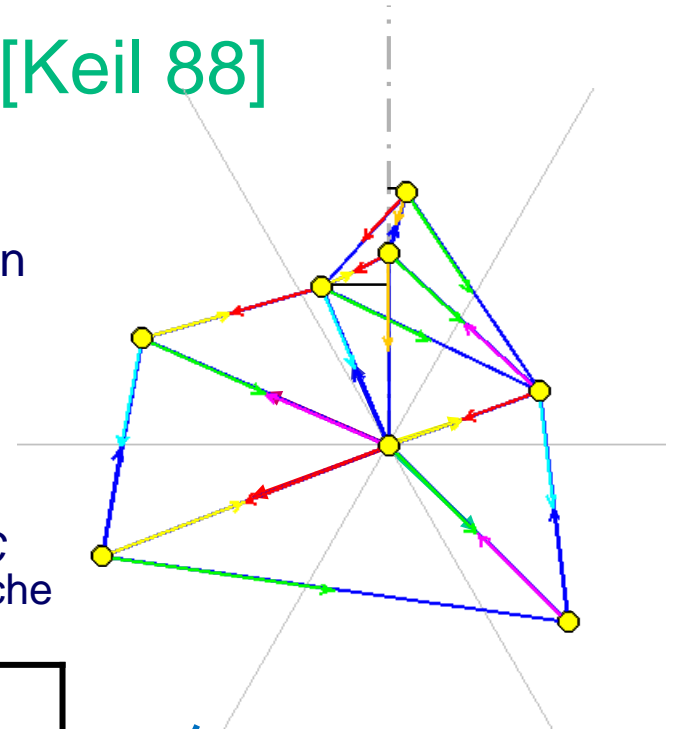
<i>pgone</i>	Borne inf	Borne sup
3 	2 [Chew 89]	2 [Chew 89]
4 	2.61	3.16 [Chew 86] 2.61 [B. Gavoille Hanusse Perković12]
5 	1.96	?
6 	2	?
$p > 6$ 	?	?
$\infty$ 	1.593 [Xia Zhang 11]	1.998[Xia 11]

# $\theta$ -graphe

[Clarkson 87][Keil 88]

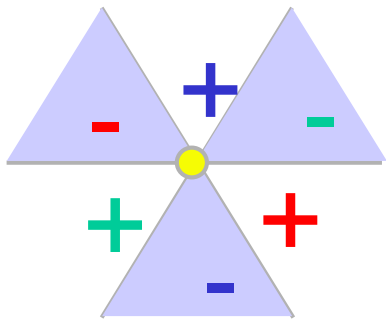


- Soit  $k \geq 2$ ,  $\theta = 2\pi/k$ .
- Le plan autour de chaque sommet  $s$  est divisé en  $k$  cônes.
- Les arêtes du  $\theta$ -Graphe sont définies de la manière suivante :  
 $\forall s \in S, \forall$  cône  $C$ .
  - Ajouter l'arête  $(s,p)$  où  $p$  est le sommet du cône  $C$  qui a le projeté sur la bissectrice de  $C$  le plus proche de  $s$ .



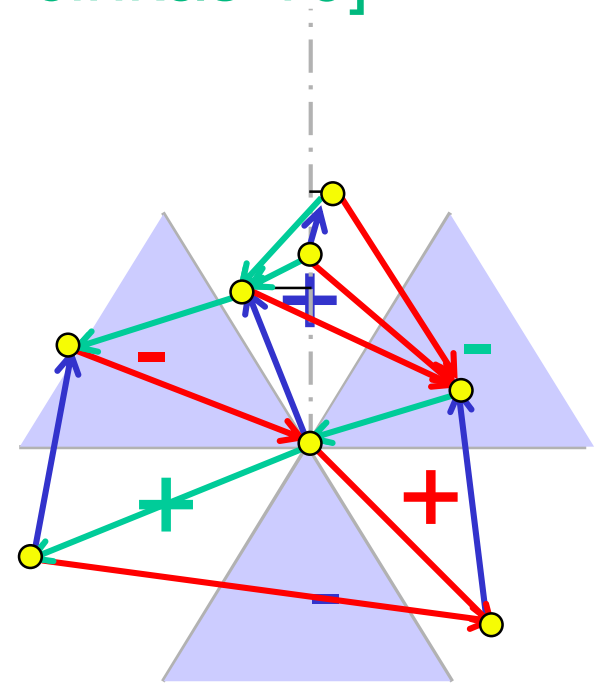
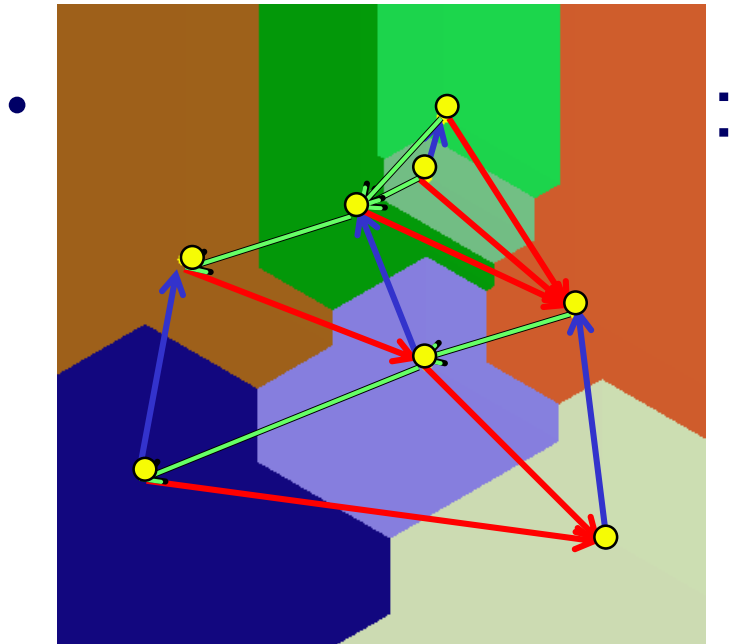
papier	k	étirement
[El Molla 09]	2,3	$\infty$
[Barba et al. 13+]	4	237
[Bose et al. 13+]	5	9.96
[B. Gavoille Hanusse Ilcinkas 10]	6	2
[Clarkson 87][Keil 88]	$\geq 7$	$\frac{1}{1-2\sin(\pi/k)}$

k	étirement
7	7.56
8	4.26
...	...
12	2.07
13	1.91



# $\frac{1}{2}\theta$ -graphe

[B. Gavoille Hanusse Ilcinkas 10]



**Thm** [B. Gavoille Hanusse Ilcinkas 10]:

$\frac{1}{2}\theta$ -Graphe  $\Leftrightarrow$  TD-Delaunay (+structure)

**Corollaire :**

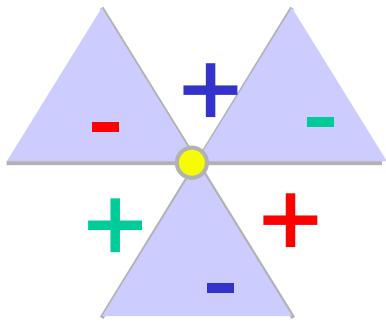
- Les  $\frac{1}{2}\theta$ -Graphes sont des 2-spanners planaires
- $\theta_6$  est un 2-spanner.

[illegible]



# Spanners planaires degré borné

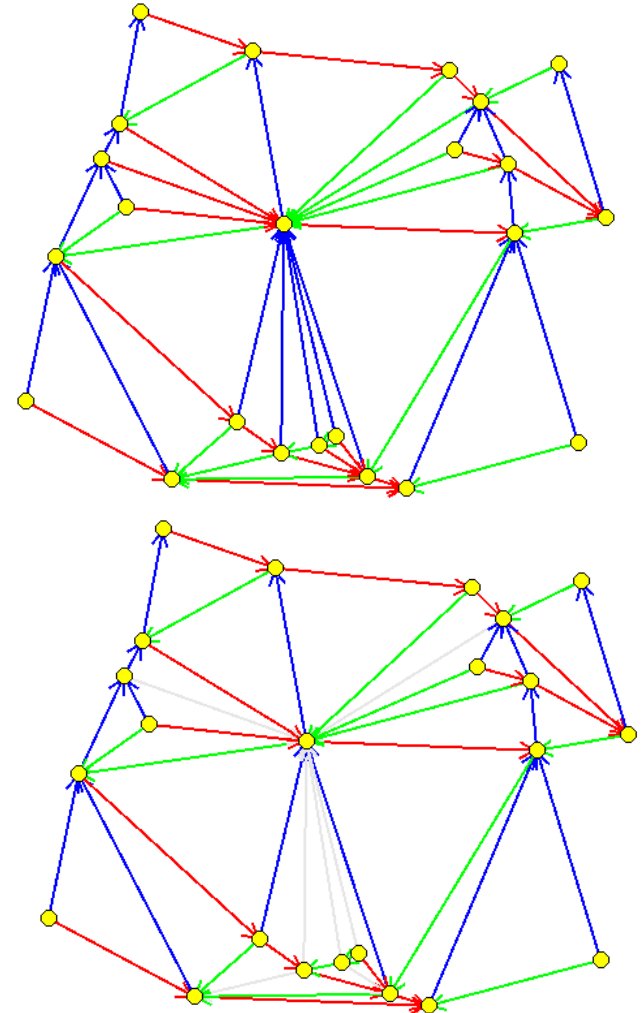
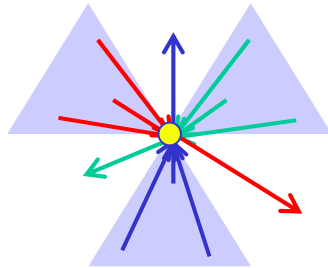
Paper	DegMax	Planar	stretch factor
Folklore	2	✗	$\infty$
[Salowe 94]	4	✗	$C > 405$
[Das and Heffernan 96]	3	✗	
[Bose <i>et al.</i> 05]	27	✓ ○	10.016
[Li and Wang 04]	23	✓ ○	7.79
[Bose <i>et al.</i> 09]	17	✓ ○	28.54
[Kanj and Perković 08]	14	✓ ○	3.53
[B. Gavoille Hanusse Perković 10]	6 12	✓ △	6
[Bose <i>et al.</i> 12]	6	✓ ○	81.66
[B. Kanj Perković Xia 14]	4	✓ □	$16.9 < C < 156$



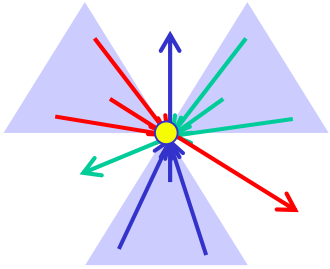
$H_{12}$ : étirement 6 et  $\deg_{\max} \leq 12$

- $H_{12}$ : pour chaque cône (-) on garde au plus 3 arêtes :
  - La plus "courte"
  - La plus à gauche
  - La plus à droite

- Degré :







# Étirement de $H_{12}$

- **Idée:** chaque arête de  $H_{ifn}$  est remplacée par un chemin 3 fois plus long dans  $H_{12}(S)$   
 $\Rightarrow H_{12}$  a un étirement  $\leq 6$ .

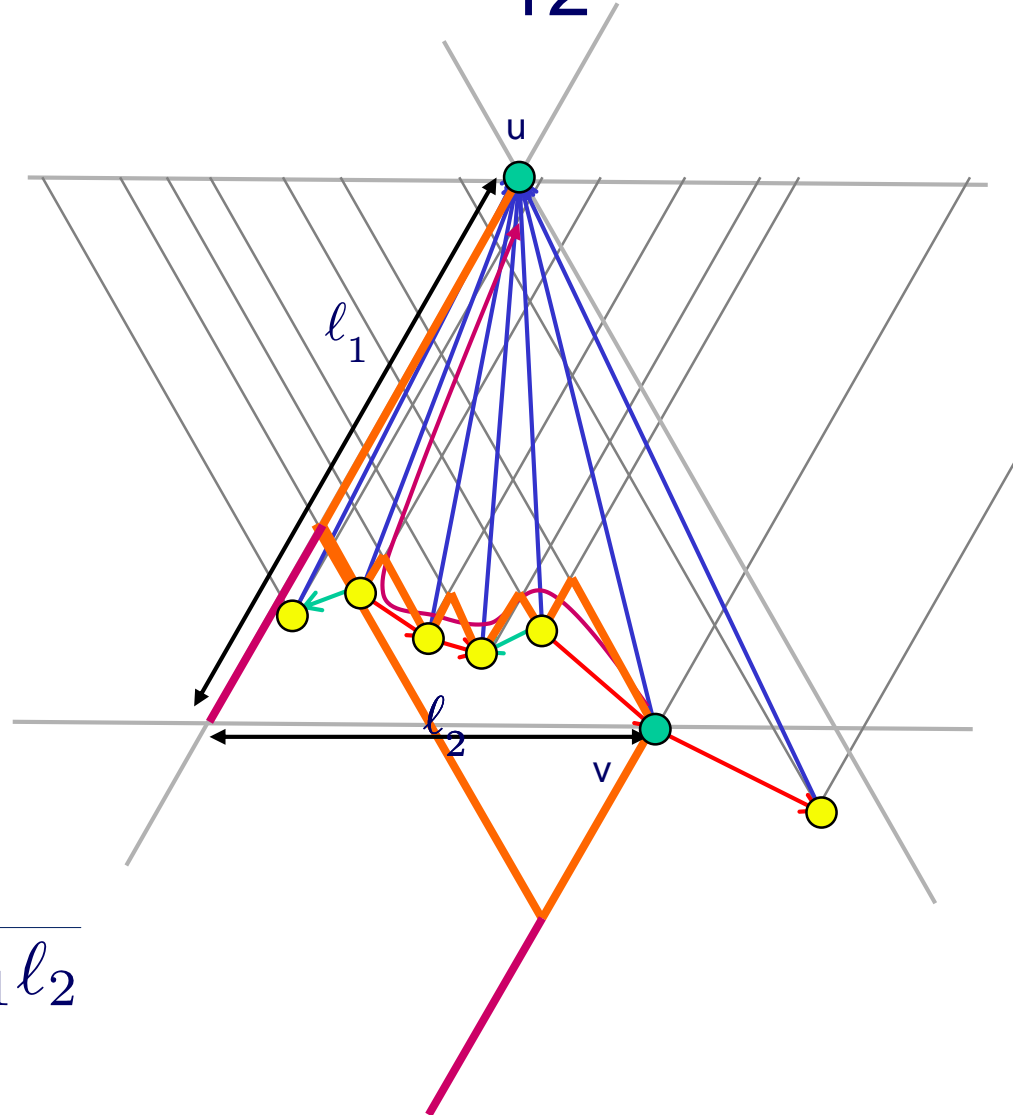
- Pour chaque arête  $uv$  supprimée, on considère le **chemin canonique** :

- Faire le tour de la face en utilisant l'**arête courte**.

$$|Path(u, v)| \leq \ell_1 + 2\ell_2$$

$$d(u, v) = \sqrt{\ell_1^2 + \ell_2^2} - \ell_1\ell_2$$

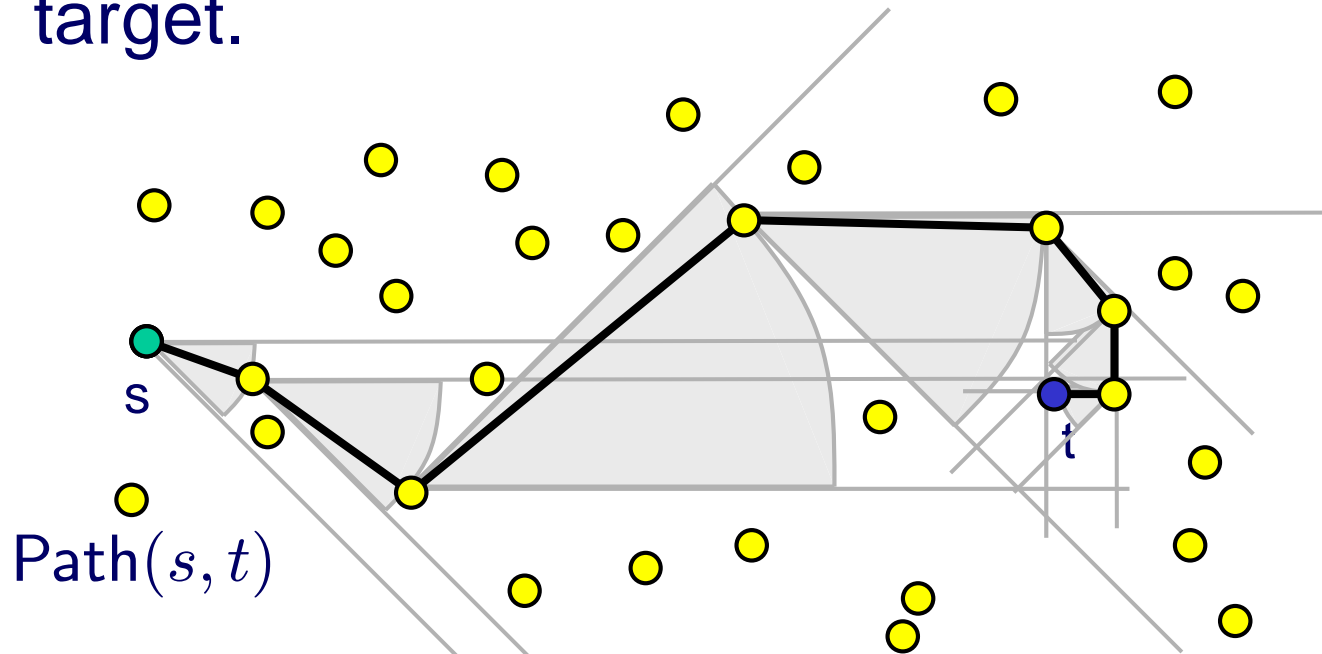
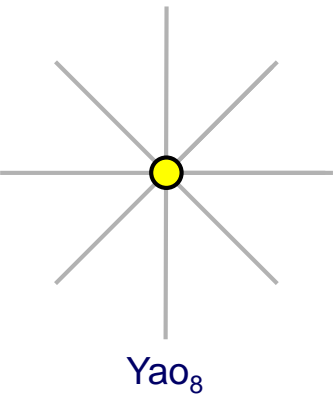
$$\frac{|Path(u, v)|}{d(u, v)} \leq 3$$





# Compass routing algorithm (*cross*)

- Compute a **path** by selecting at each step the neighbour in the direction of the target.



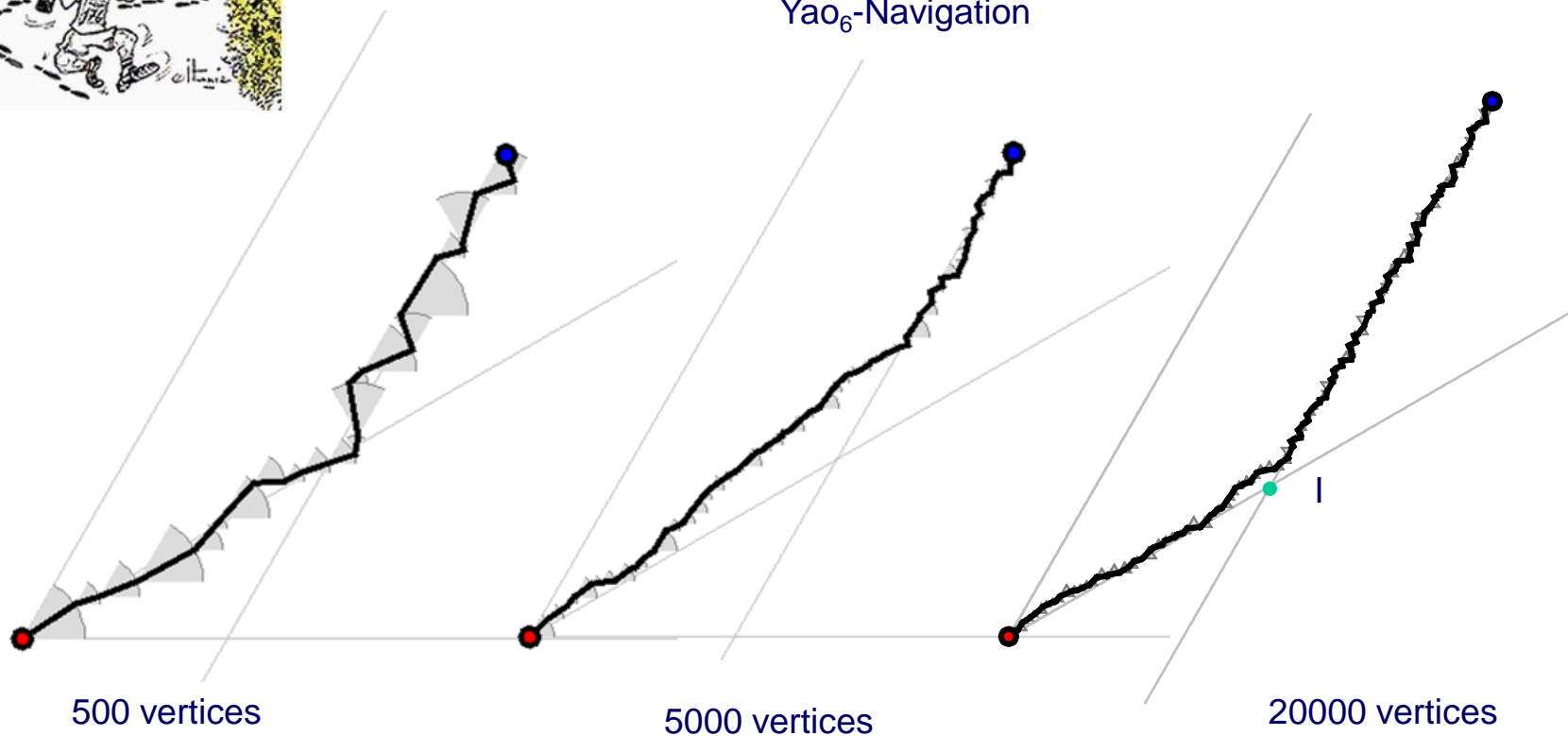
$$|\text{Path}(s, t)| \leq \frac{|st|}{1 - 2 \sin(\Theta/2)}$$

Yao-8



# When the point set becomes big ...

Yao<sub>6</sub>-Navigation



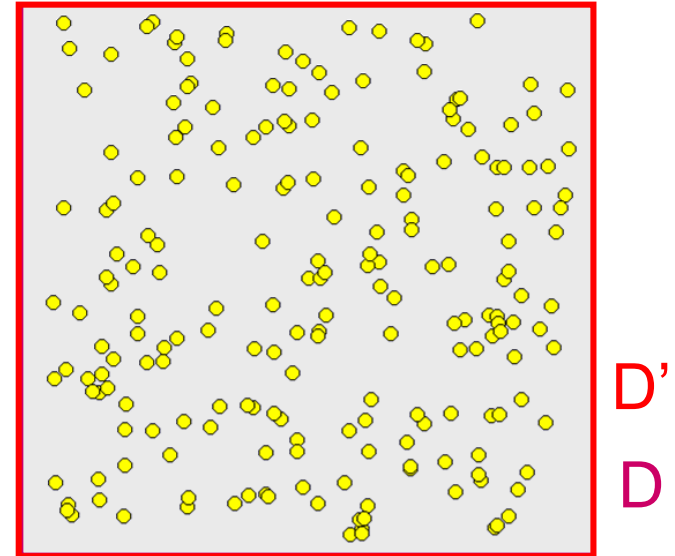
$$Path(s, t) \rightarrow [s, I] \cup [I, t]$$

$$|Path(s, t)| \rightarrow 1.047||s, I|| + 1.209||I, t||$$

$$\text{For } \theta = \pi/3, \frac{\theta/2}{\sin(\theta/2)} \approx 1.047 \text{ and } \frac{\theta}{\sin(\theta)} \approx 1.209$$

# Probabilistic model

- **D: domain**
  - a bounded simply connected open of  $\mathbb{C}$
  - (e.g. unit square)
- **S: Poisson Point Process of intensity  $n$**



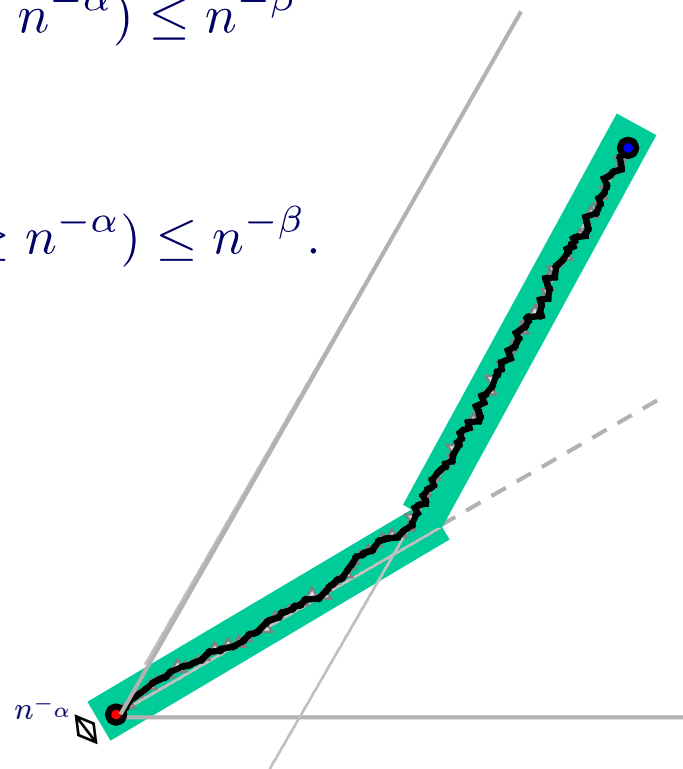
# [B. Marckert 12]

Let  $\mathbf{Q}(s, t) = \frac{\theta/2}{\sin(\theta/2)} \|s, I\| + \frac{\theta}{\sin(\theta)} \|I, t\|$  and  $\Gamma(s, t) = [s, I] \cup [I, t]$ .  
For any  $\alpha \in (0, 1/8)$ , any  $\beta > 0$ , for  $n$  large enough

$$\mathbb{P}\left(\sup_{(s,t) \in D'} |\text{Path}(s, t)| - \mathbf{Q}(s, t)| \geq n^{-\alpha}\right) \leq n^{-\beta}$$

and

$$\mathbb{P}\left(\sup_{(s,t) \in D'} d_H([\text{Path}(s, t)], \Gamma(s, t)) \geq n^{-\alpha}\right) \leq n^{-\beta}.$$

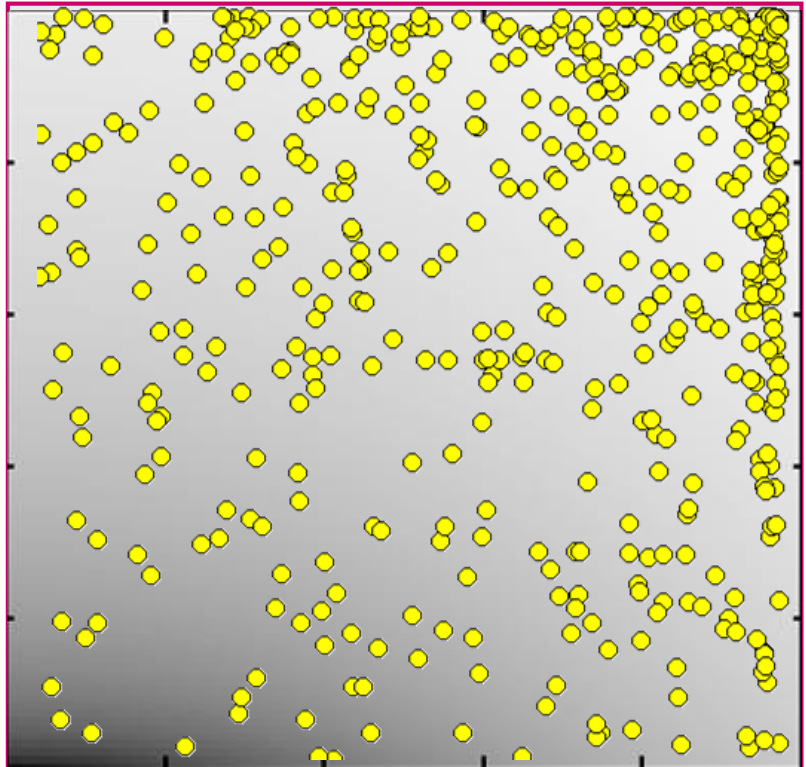


# Non-uniform distribution

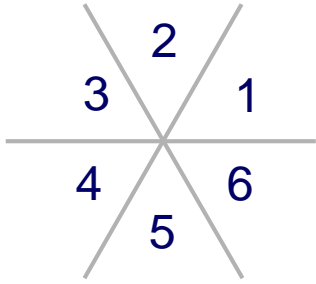
- $f$  : smooth **density** function
  - Lipschitzian  $|f(p)-f(q)| < b|p-q|$
  - $f$  strictly positive on  $D$
  - (e.g.  $f(p)=1$  or  $f(p)=2-|p|^2$ )

$$\forall D_1 \subset D : |S \cap D_1| \sim \text{Poisson}(n \cdot \|D_1\|)$$

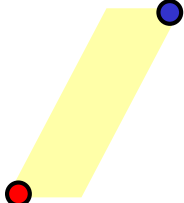
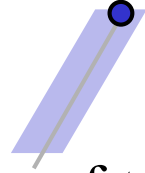
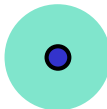
$$\forall D_1 \subset D : |S \cap D_1| \sim \text{Poisson}(n \cdot \int_{D_1} f(t) dt)$$

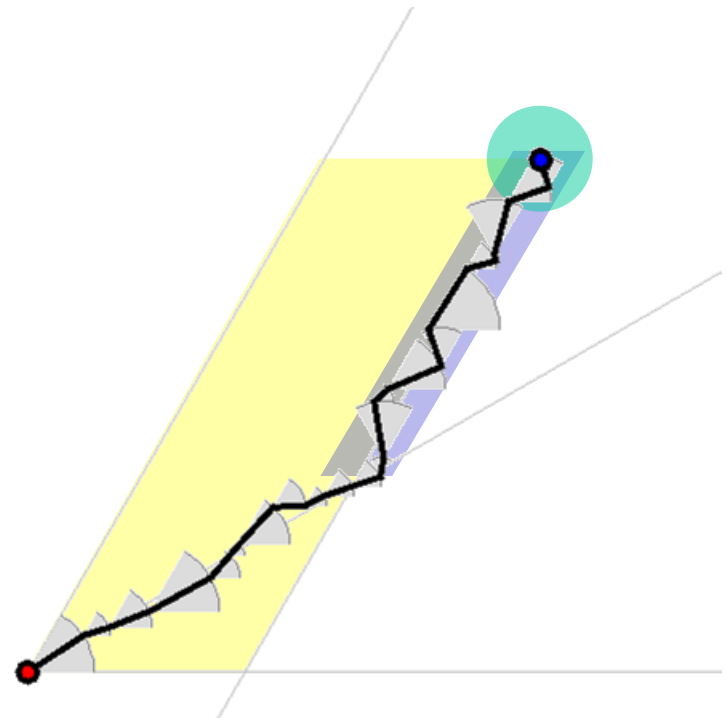


**D**



# Decomposition of the trajectory in 3 phases

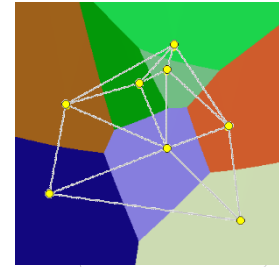
- Phase 1: 
  - $\Rightarrow$  only steps of type 1.
- Phase 2: 
  - Only steps of type 1 and 2
- Phase 3: 
  - Final approach
  - Steps of any type



# Conclusion

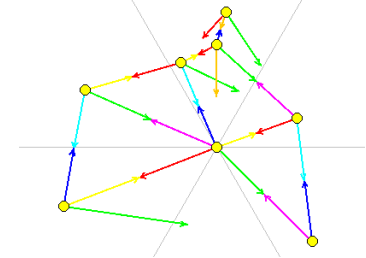
- Triangulations de  $\ast$ -Delaunay

Worst case analysis



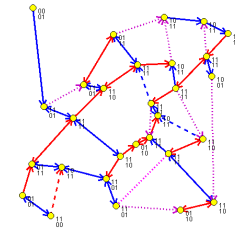
- Théta-Graphes

Worst case analysis



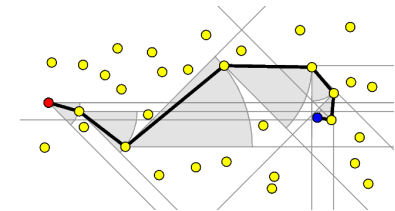
- Spanners de degré borné

Worst case analysis



- Routage dans les Théta-Graphes

Random case analysis





**Merci pour votre attention**