Enumeration of rectangulations

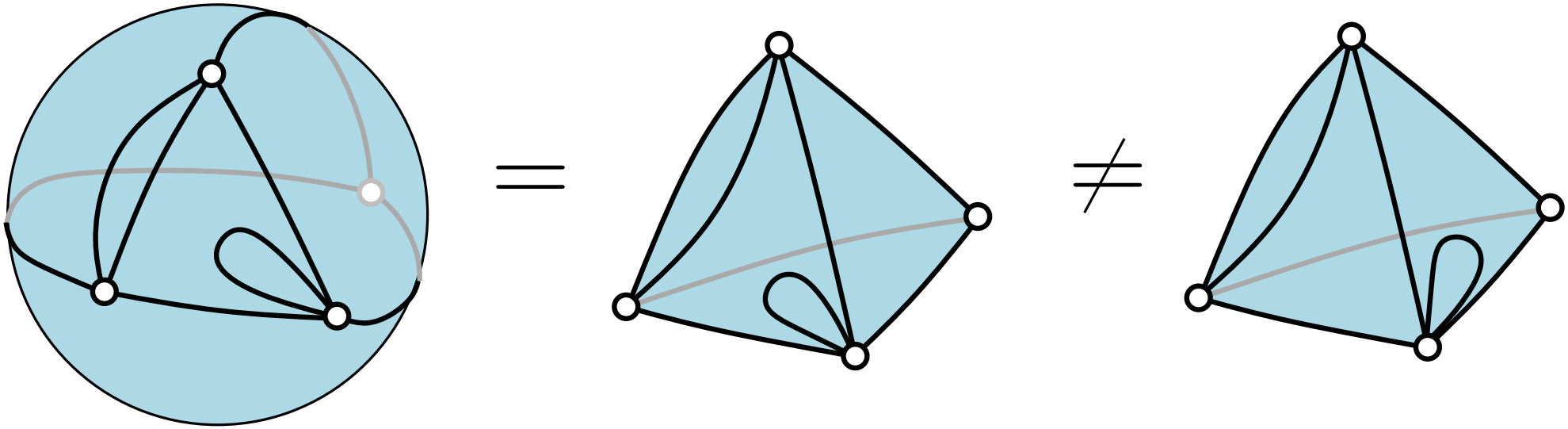
Éric Fusy (LIGM, Univ. Gustave Eiffel)

Joint work with Erkan Narmanli and Gilles Schaeffer

Séminaire combinatoire IRIF

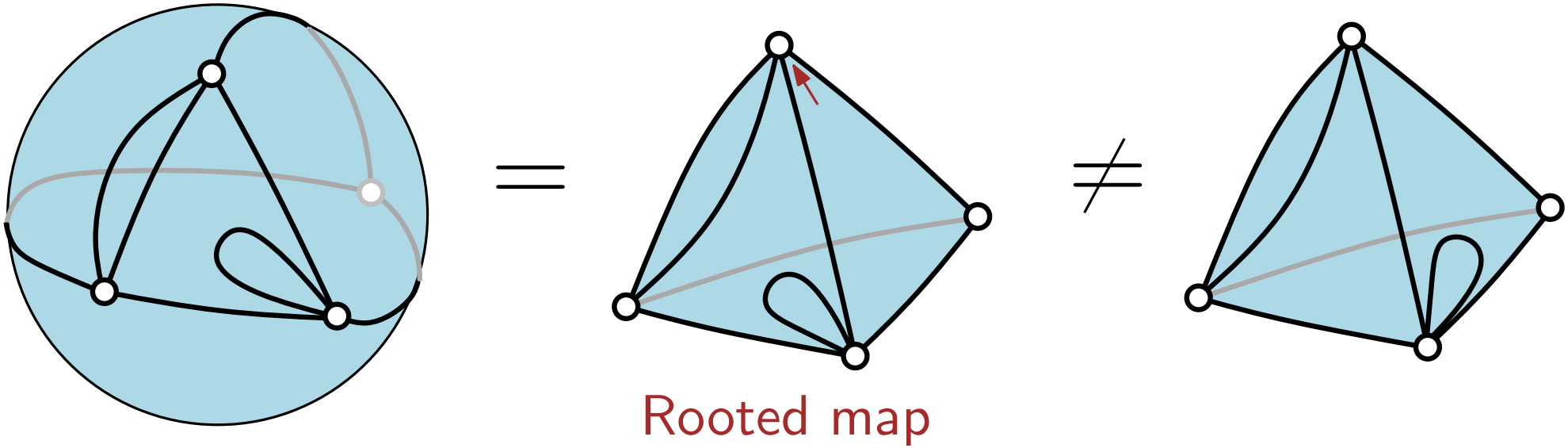
Planar maps

Def. Planar map = connected graph embedded on the sphere



Planar maps

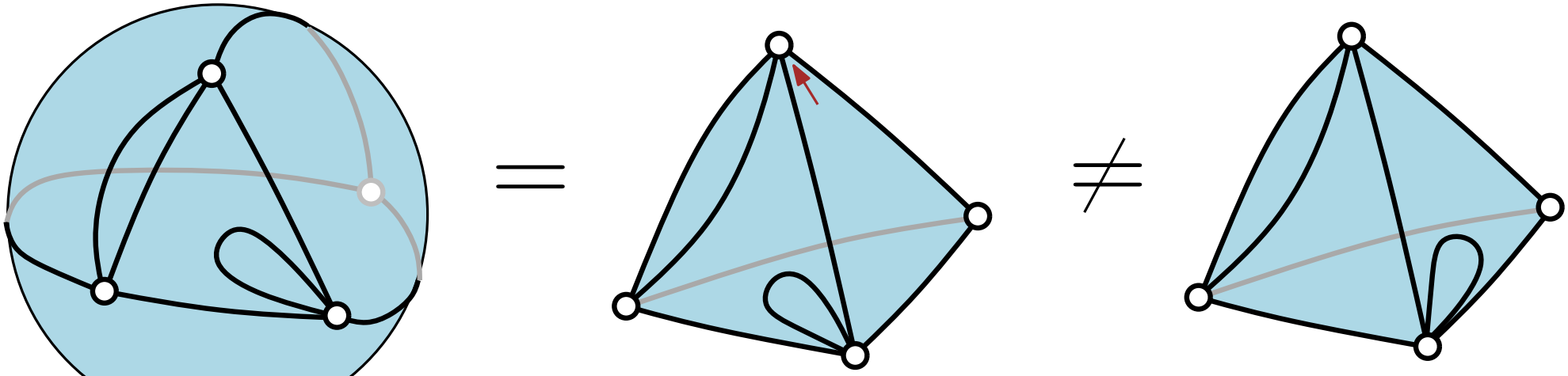
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Rooted map
= map with marked corner

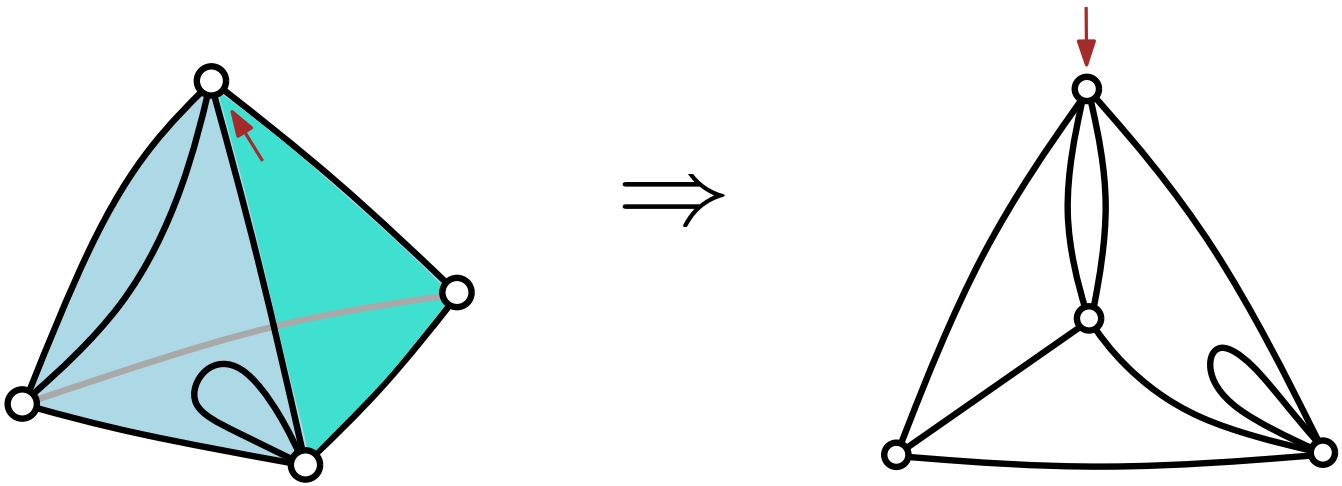
Planar maps

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Easier to draw in the plane (choosing root-face to be the outer face)



Universality properties for planar maps

- Nice counting formulas for many natural families [Tutte'60s]

  e.g. $\#$ rooted maps n edges $= \dfrac{2}{n+2} 3^n \mathrm{Cat}_n$

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- Universal scaling limit (Brownian sphere) for random planar maps

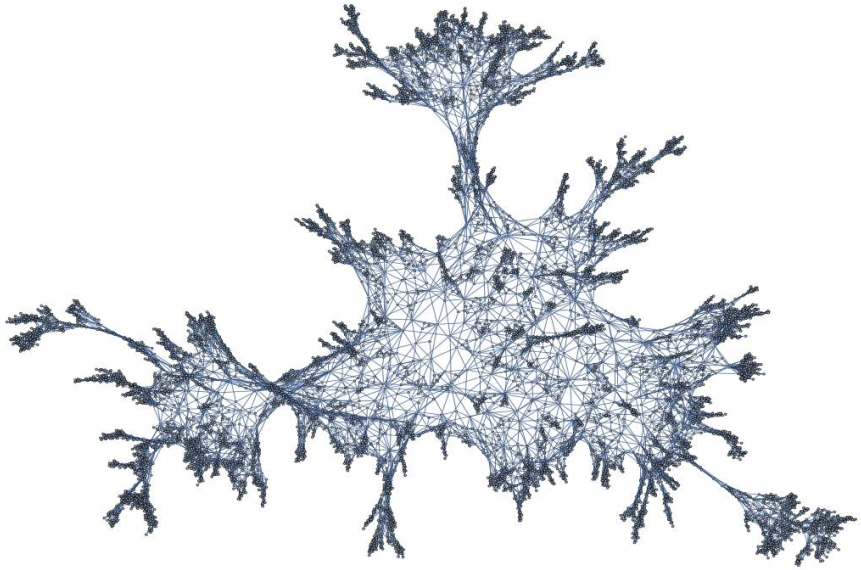
  (rescaling distances by $n^{1/4}$)

  [Chassaing,Schaeffer'04]

  [Le Gall'13, Miermont'13]

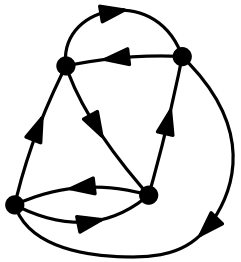
  Link to Liouville Quantum Gravity

  [Miller-Sheffield'12]



© Nicolas Curien

Decorated planar maps

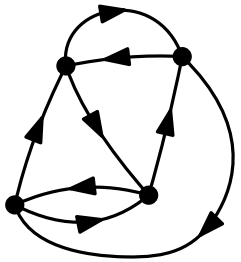


4-regular map
+ Eulerian orientation

Decorated planar map = planar map + structure

(Ising model, proper coloring, Potts model,
spanning tree, spanning forest, specific orientations,...)

Decorated planar maps



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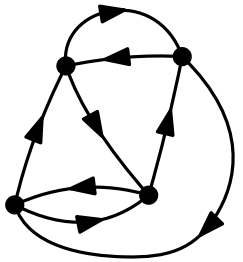
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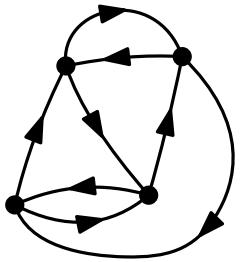
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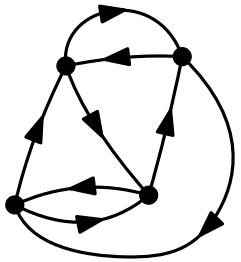
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conjectural scaling limits & bounds on magnitude of typical distances

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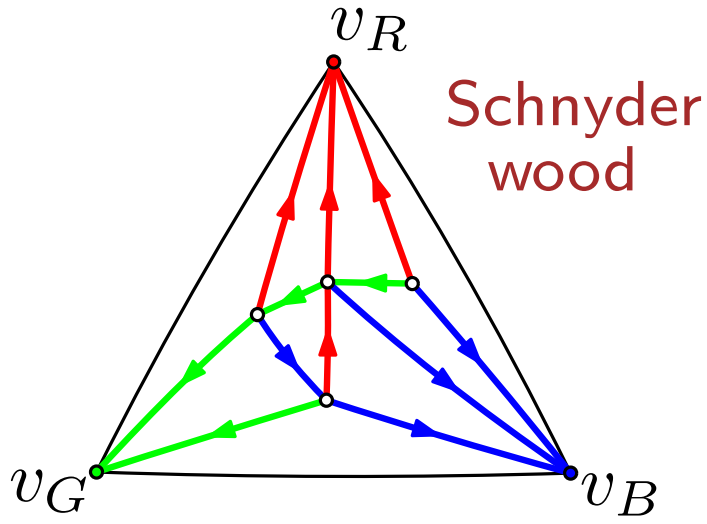
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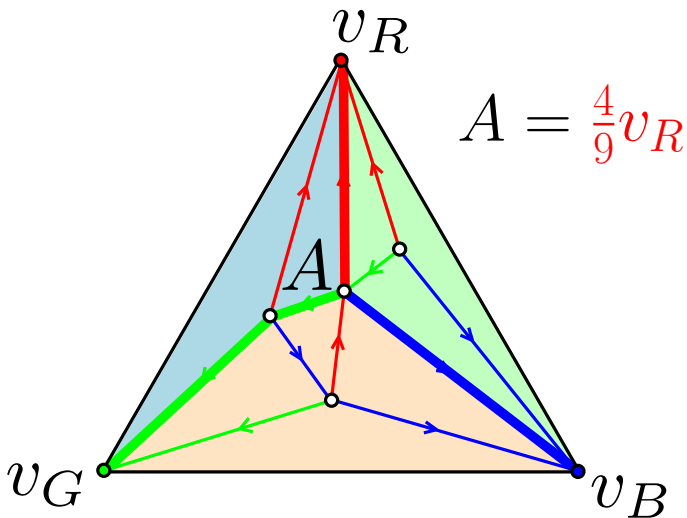
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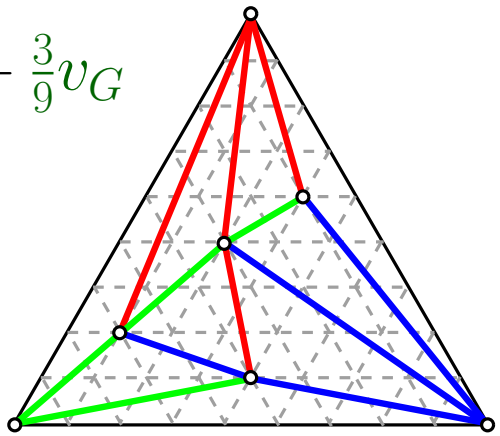
- Some of these structures give nice geometric representations of maps



Schnyder wood



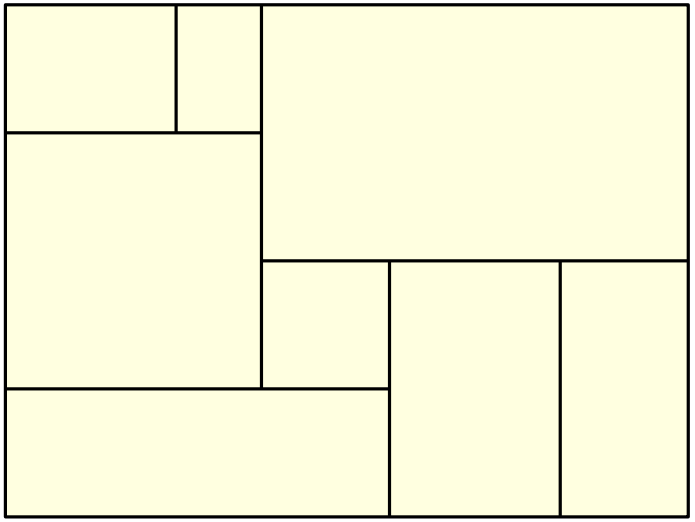
$A = \frac{4}{9}v_R + \frac{2}{9}v_B + \frac{3}{9}v_G$



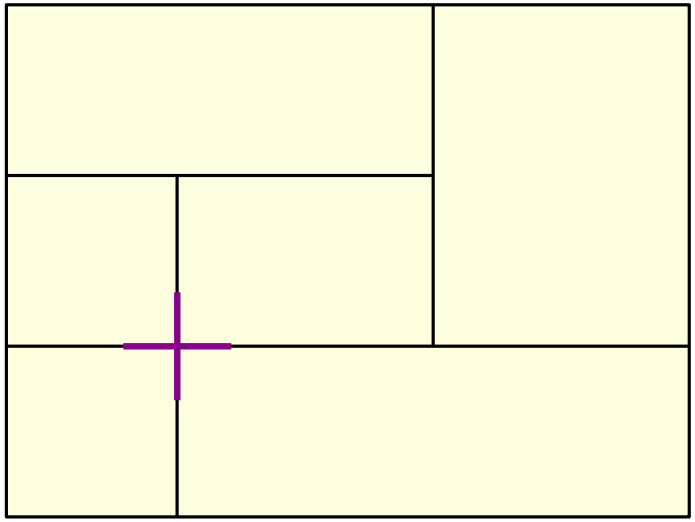
Rectangulations

Rectangulation = tiling of a rectangle by rectangles

Called "generic" if no 



Generic

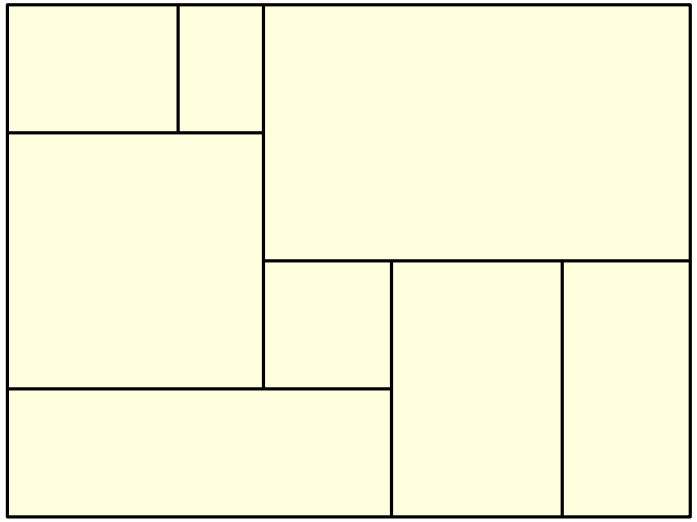


Not generic

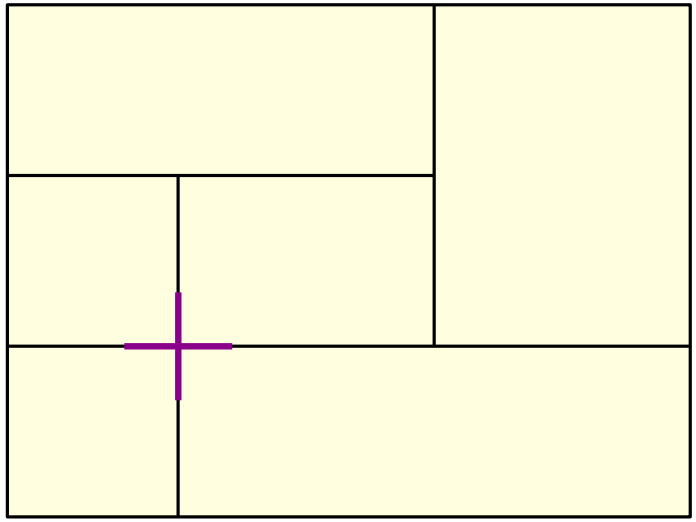
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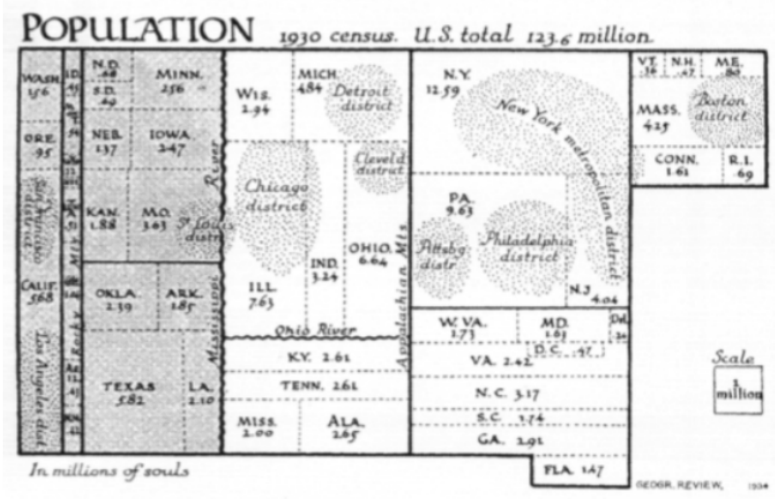


Not generic

used in "cartogram" representations

[van Kreveld-Speckmann'04]

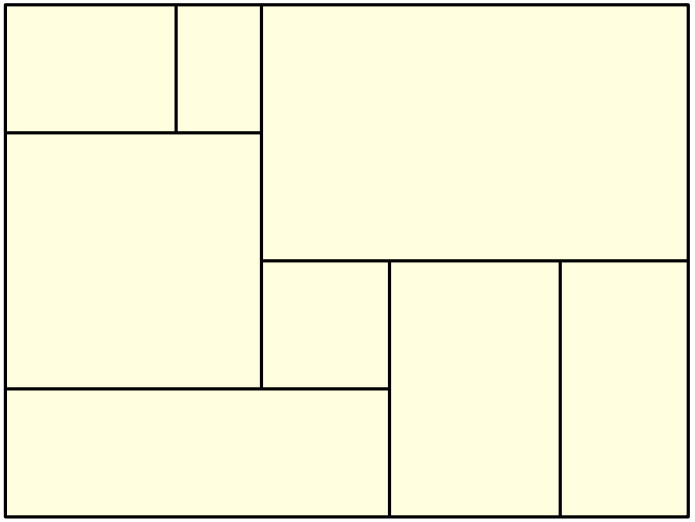
[Eppstein-Mumford-Speckmann-Verbeek'12]



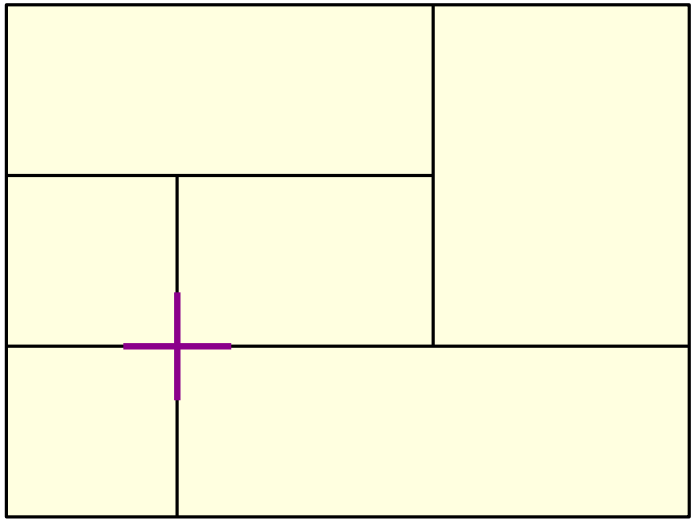
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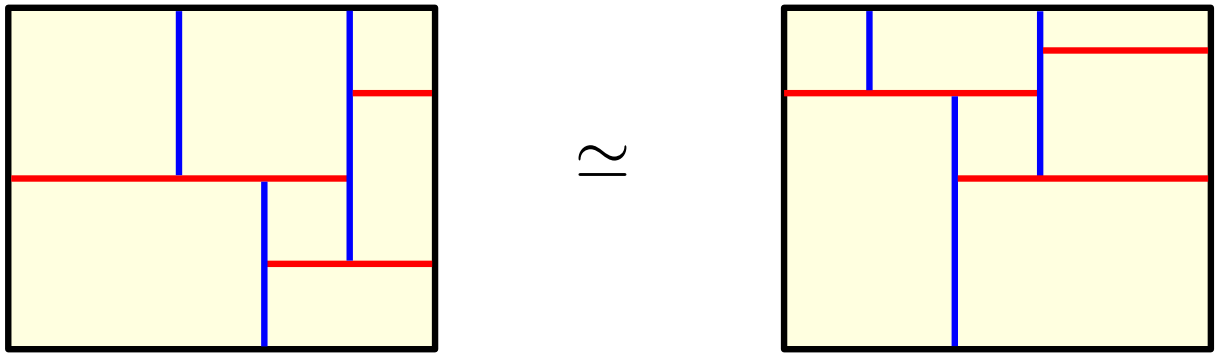
Not generic

This talk:

- Link to decorated planar maps & bijections to walks
- Exact enumeration
- Asymptotic enumeration

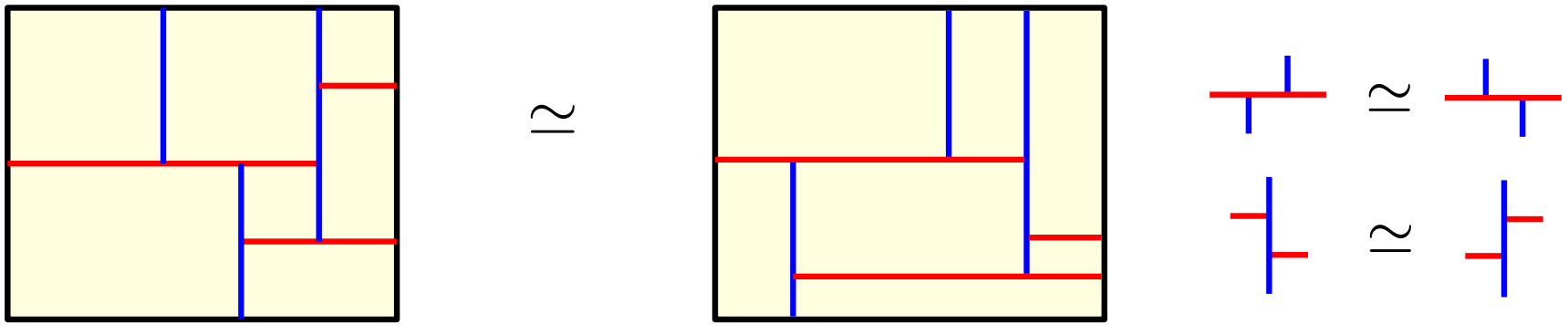
Two types of equivalences for rectangulations

Strong



(order of contacts along each maximal segment is preserved)

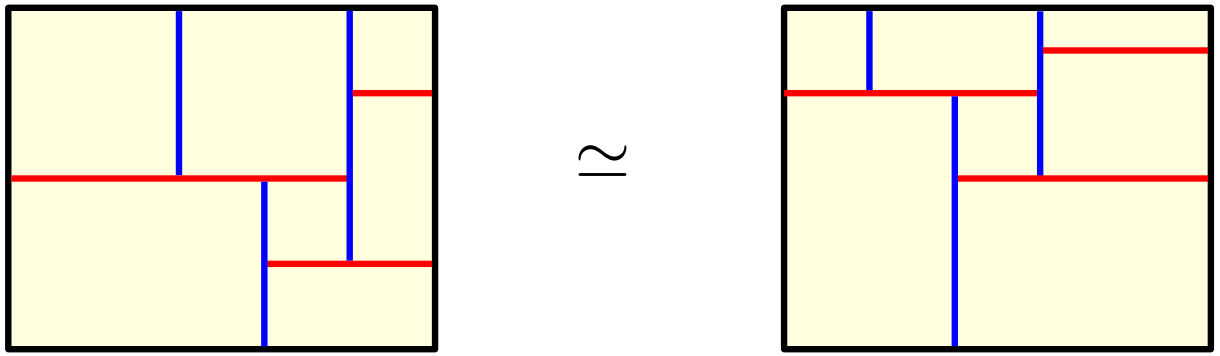
Weak



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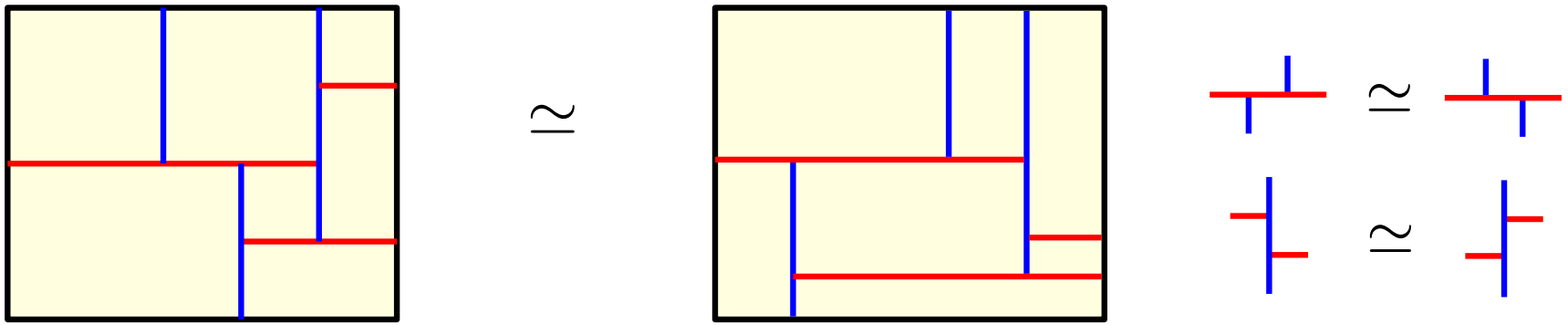
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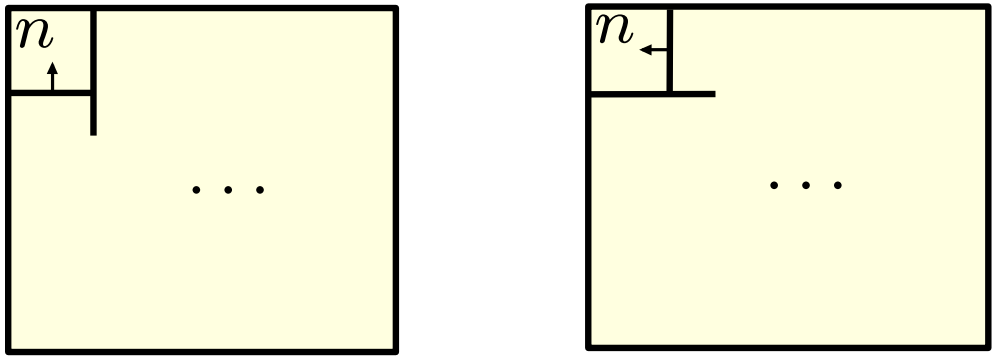
$w_n =$ # weak equivalence classes with n regions

$s_n =$ # strong equivalence classes with n regions

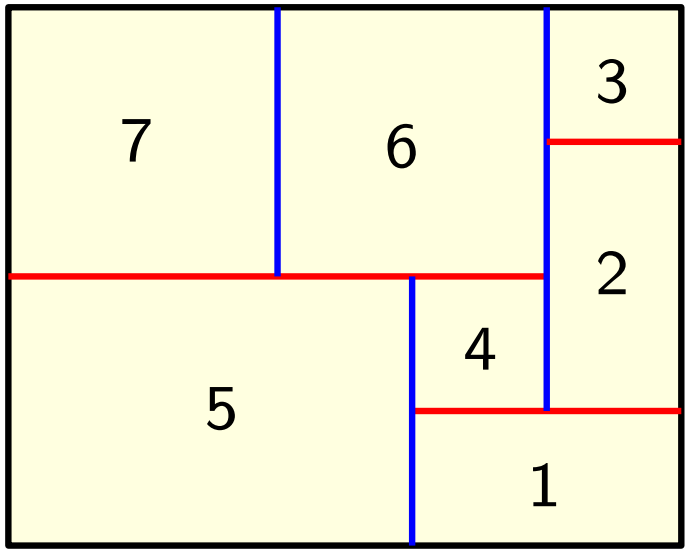
Weak equivalence class: shelling order

[Ackerman, Barequet, Pinter'06]

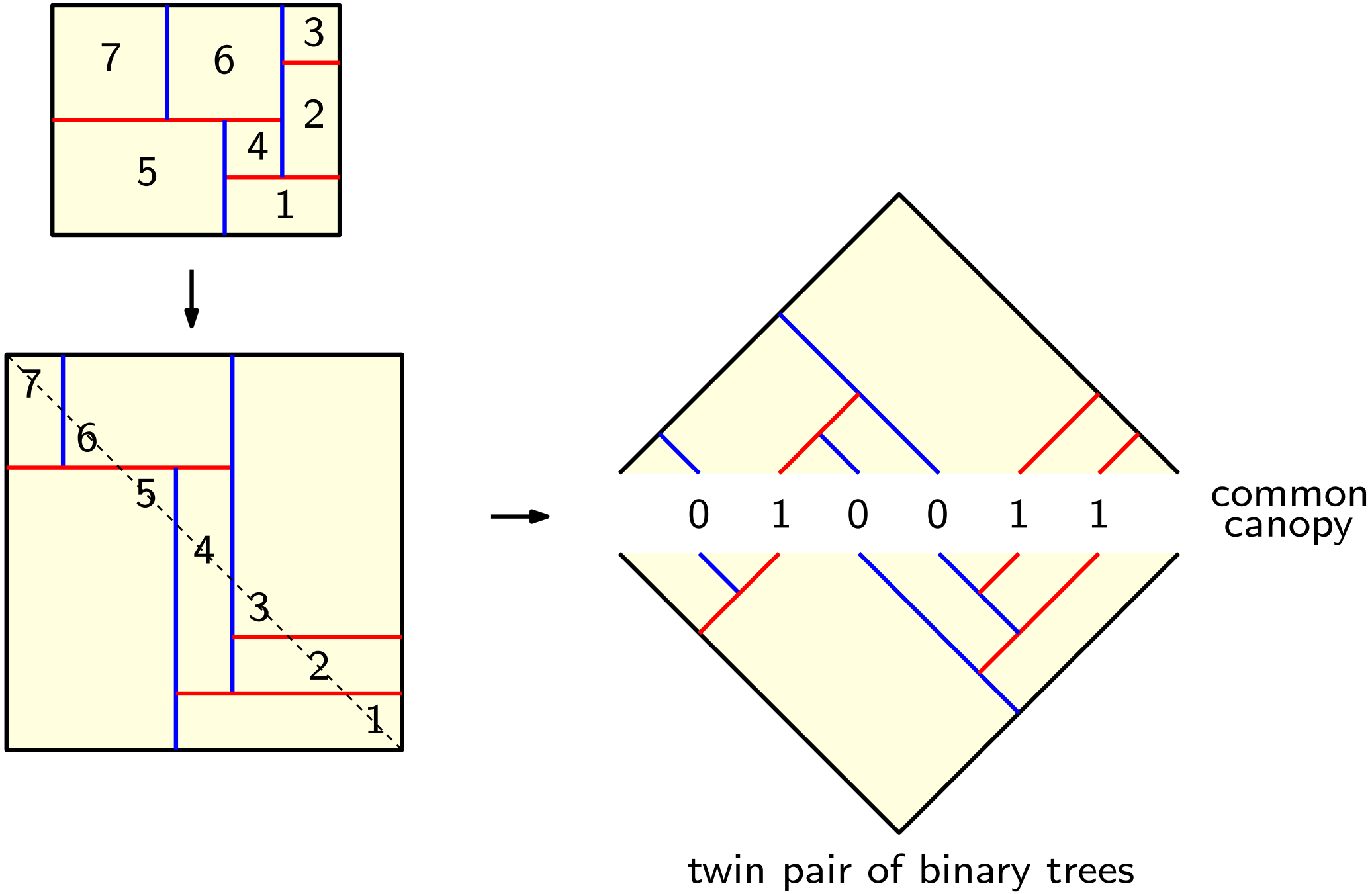
Contract top-left region:
two cases



$\Rightarrow$ shelling order on regions

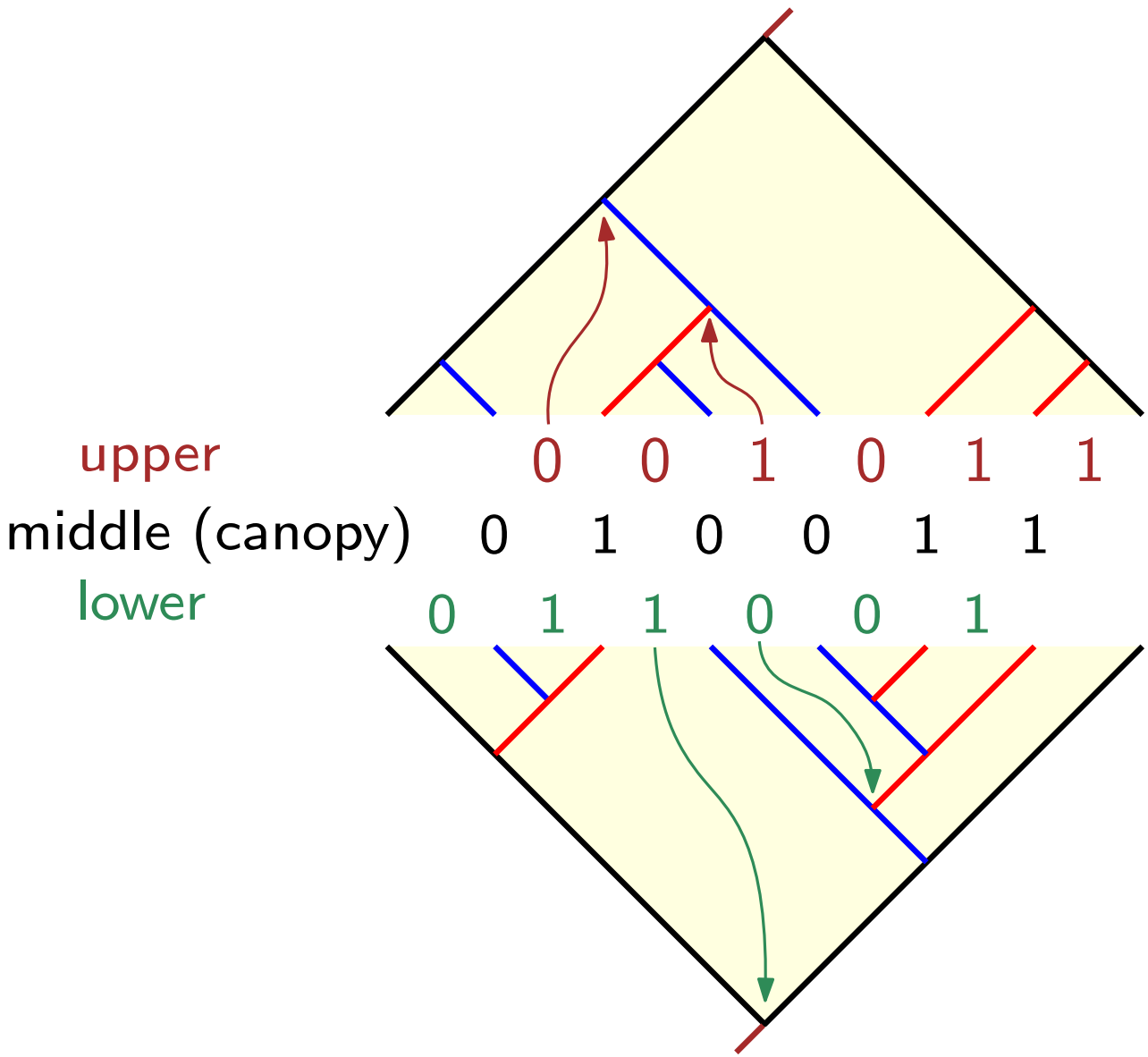


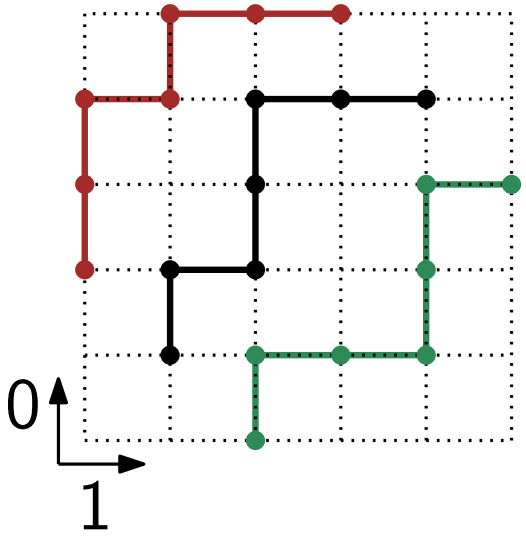
Diagonal representation



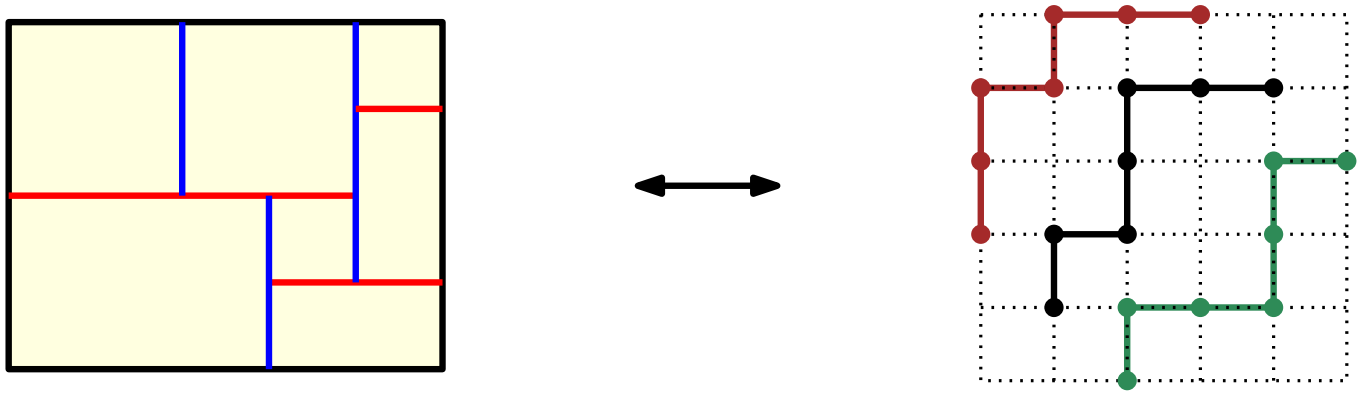
Encoding by a triple of walks

[Dulucq,Guibert'98]





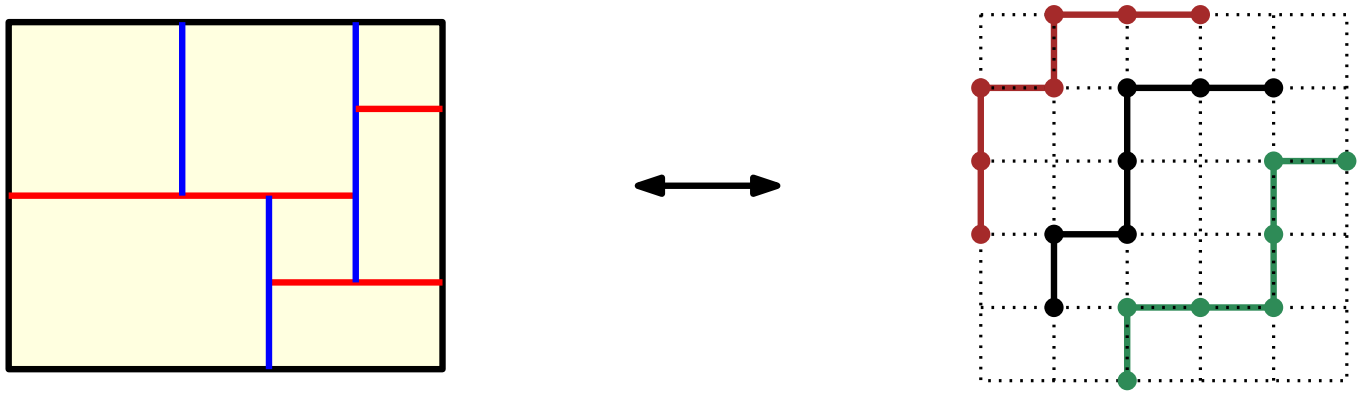
Baxter numbers and Baxter families



Gessel-Viennot $\Rightarrow$ $$w_n = \frac{2}{n(n+1)^2} \sum_{r=0}^{n-1} \binom{n+1}{r} \binom{n+1}{r+1} \binom{n+1}{r+2}$$ Baxter numbers

$$w_n \sim \frac{2^5}{\pi\sqrt{3}} 8^n n^{-4}$$

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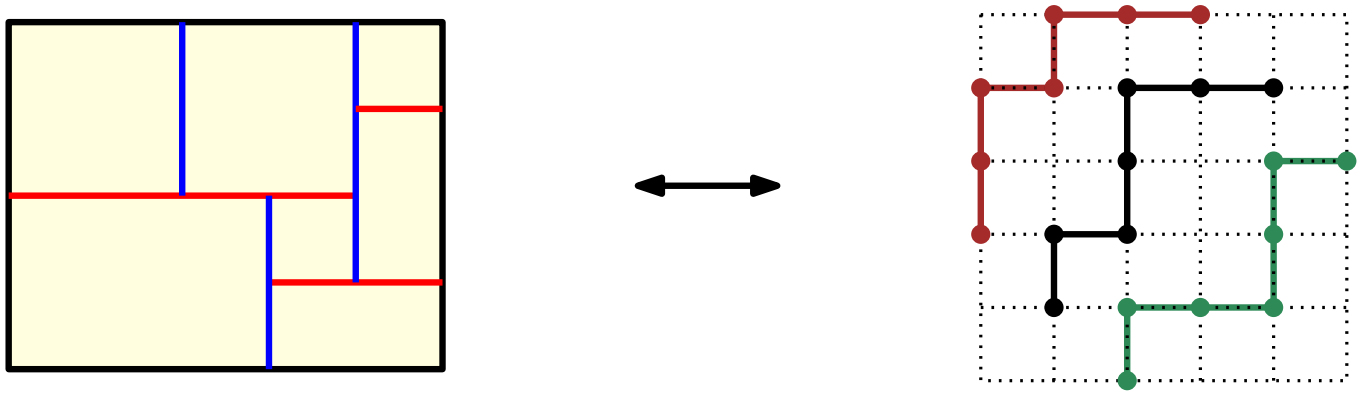
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Baxter families are families counted by Baxter numbers
among which Baxter permutations, plane bipolar orientations, ...

Various bijections relating these families (common generating tree)

[Dulucq-Guibert'98, Chow-Erikkson-Fan'05, Ackerman-Barequet-Pinter'06, F-Poulalhon-Schaeffer'07
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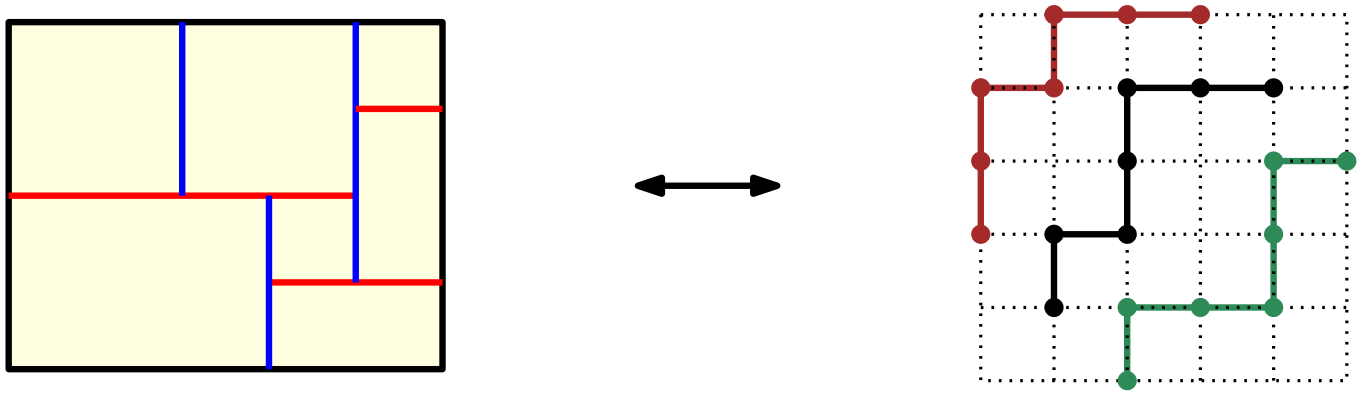
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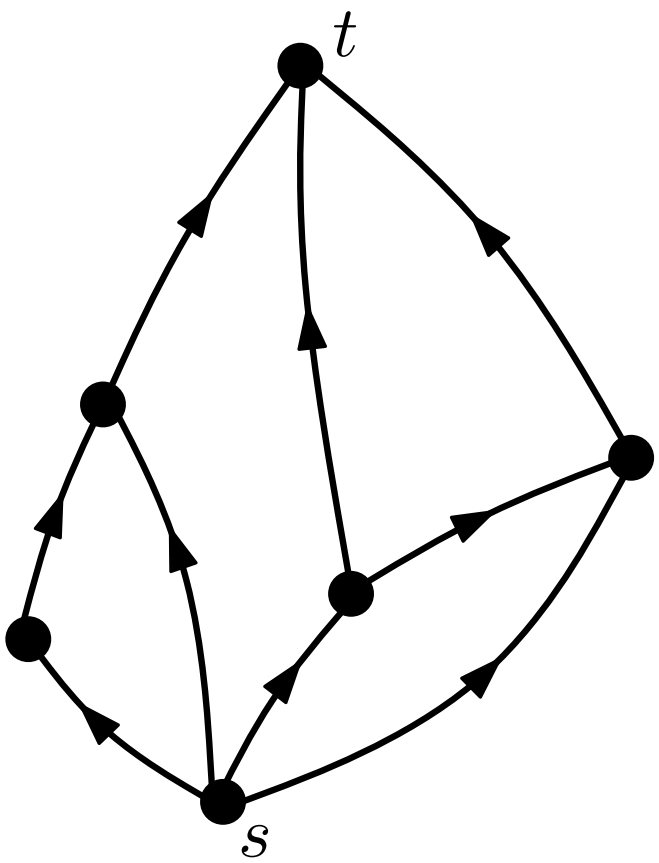
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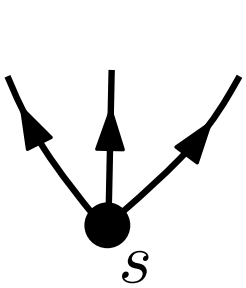
Link to weak order on permutations: [Reading'04,12]
mapping $\mathfrak{S}_n \to \mathcal{R}_n$
grouping permutations by rectangulation gives a lattice congruence

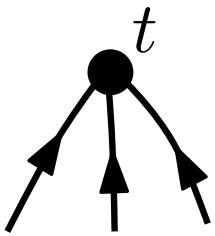
Plane bipolar orientations

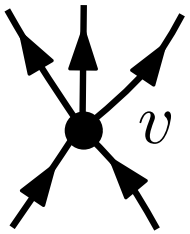


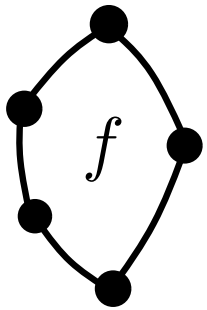
Acyclic orientation on planar map
with single min and single max
both incident to the outer face

Plane bipolar orientations $\Leftrightarrow$ local conditions

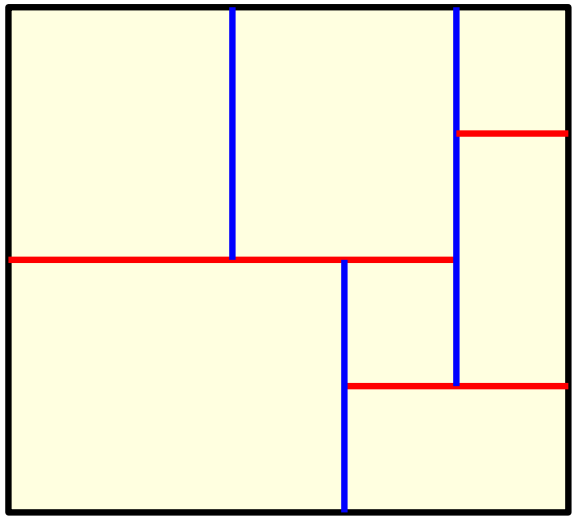


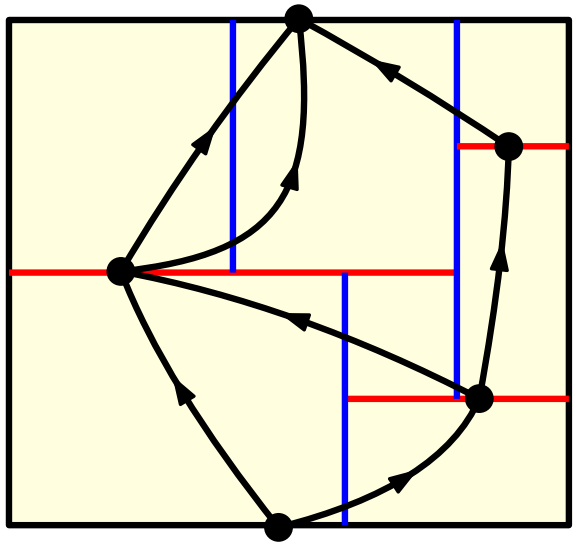


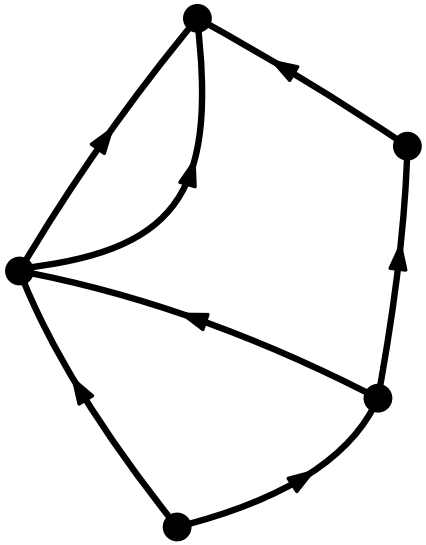


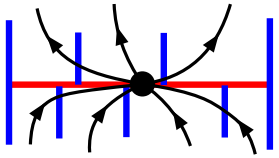


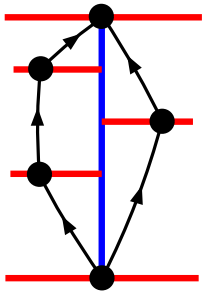
Bijective link with weak rectangulations



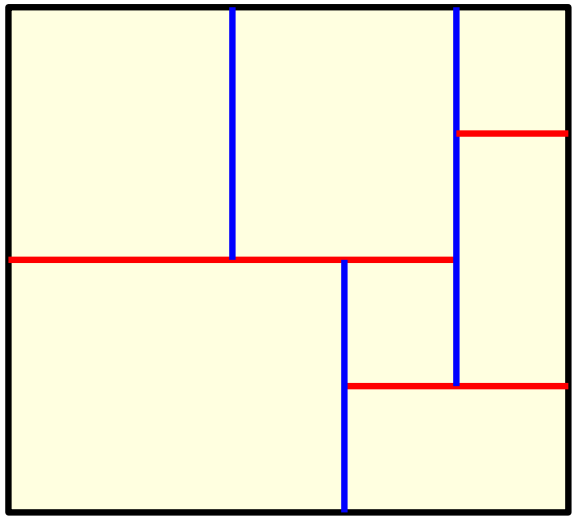


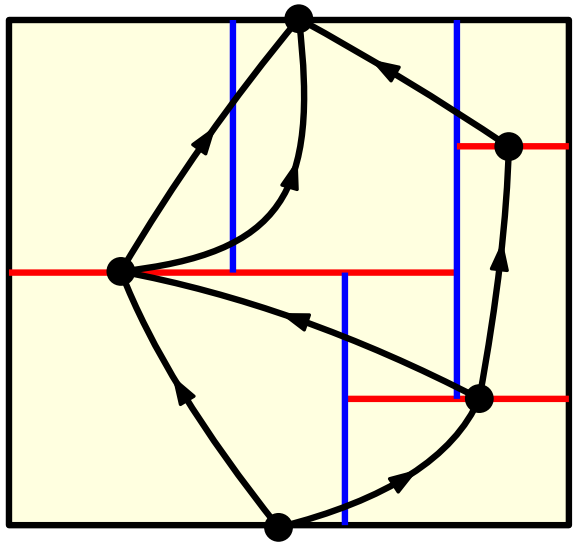


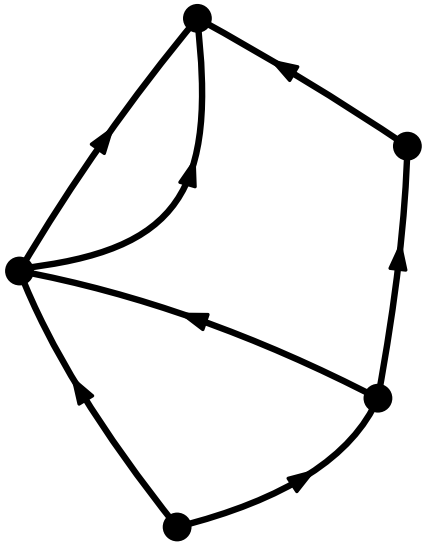


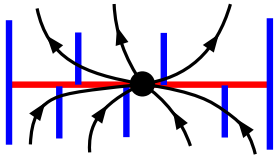


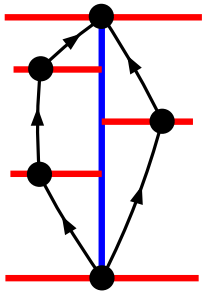
Bijective link with weak rectangulations





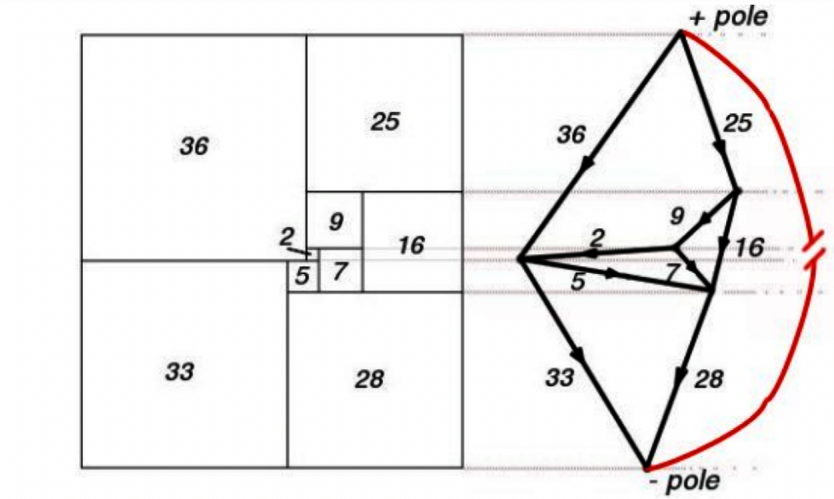






Correspondence used in problem "squaring the square"

[Brooks, Smith, Stone, Tutte'40]

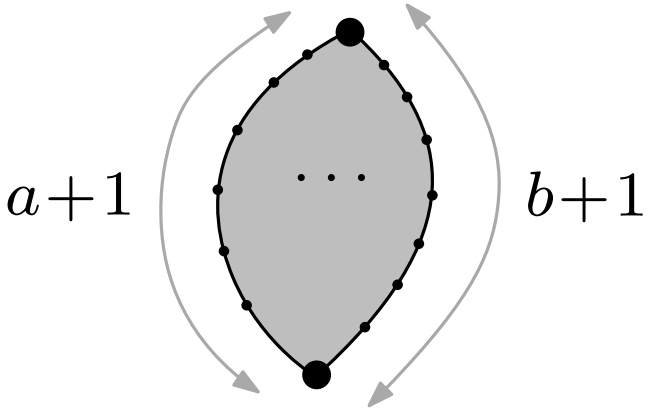


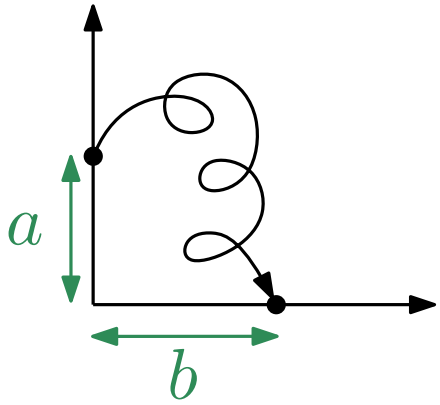
(www.squaring.net/history_theory/gfx/figure73.jpg)

A more precise walk-encoding: the KMSW bijection

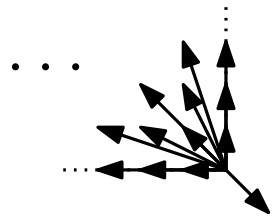
[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations $\longleftrightarrow$ "Tandem walks" in the quadrant





step-set

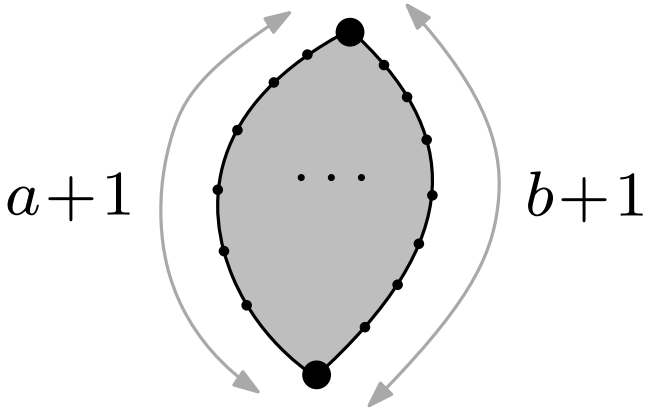


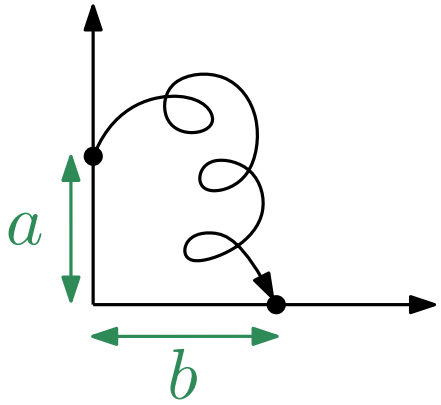
SE $\cup \{(-i, j), i, j \geq 0\}$

A more precise walk-encoding: the KMSW bijection

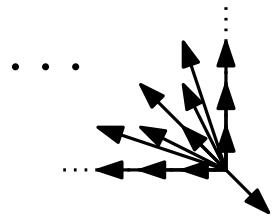
[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant



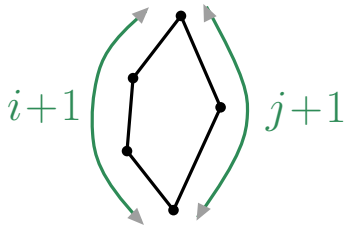


step-set



SE $\cup \{(-i,j), i,j \geq 0\}$

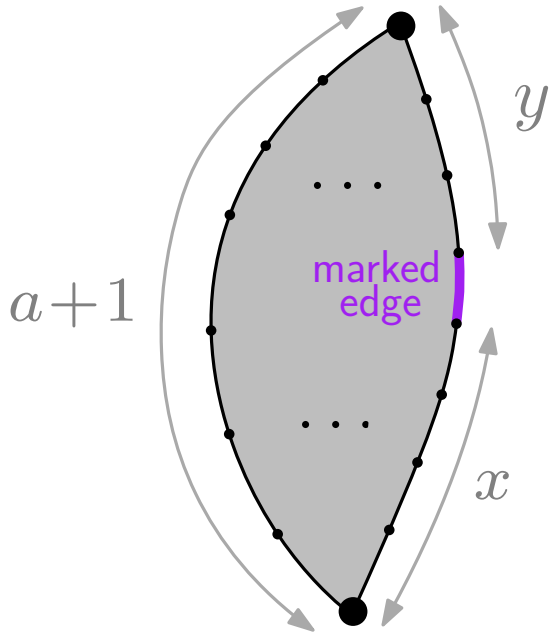
n edges ⟷ length $n-1$

face  ⟷ face-step $(-i,j)$

non-pole vertex ⟷ SE step

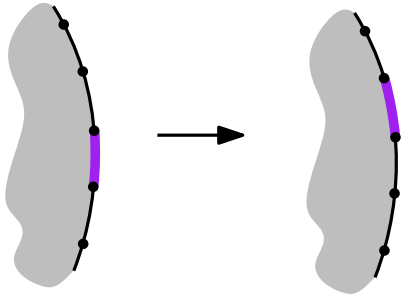
The KMSW bijection

Orientation is built step by step from the walk,

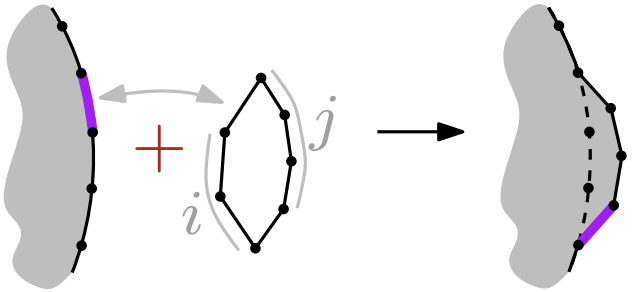


orientation
currently built

add $(1,-1)$

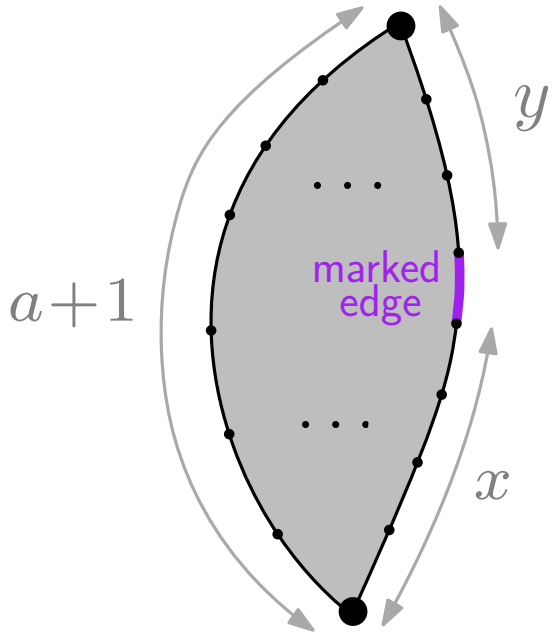


add $(-i,j)$
(face-step)



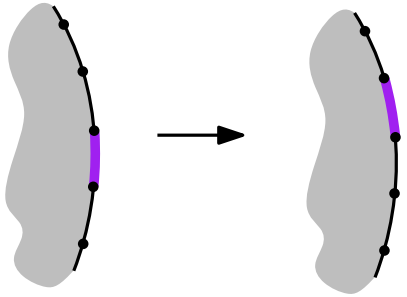
The KMSW bijection

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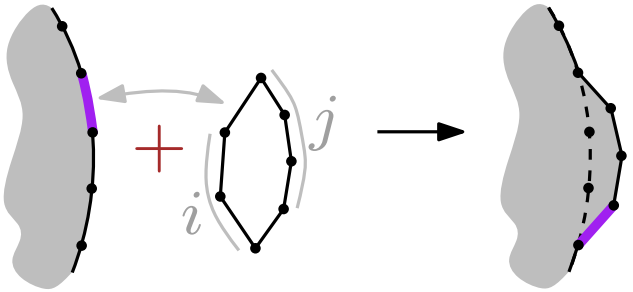


orientation
currently built

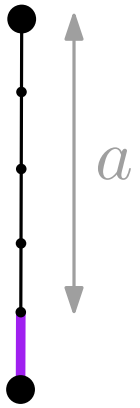
add $(1,-1)$



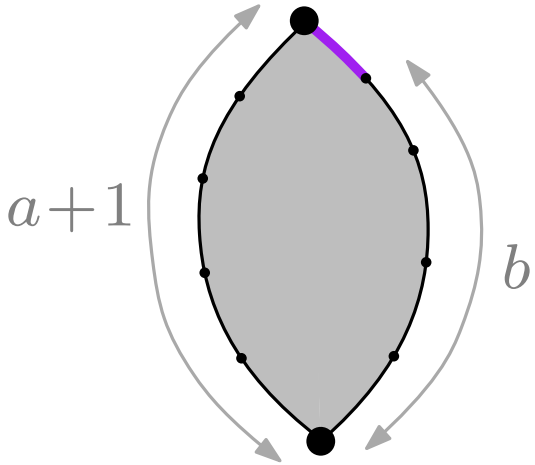
add $(-i,j)$
(face-step)



Starts with

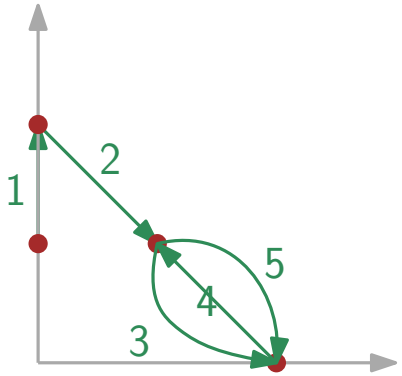


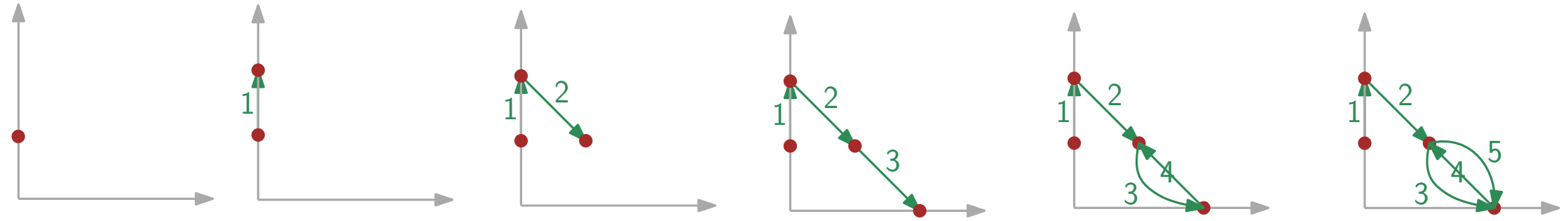
Ends with

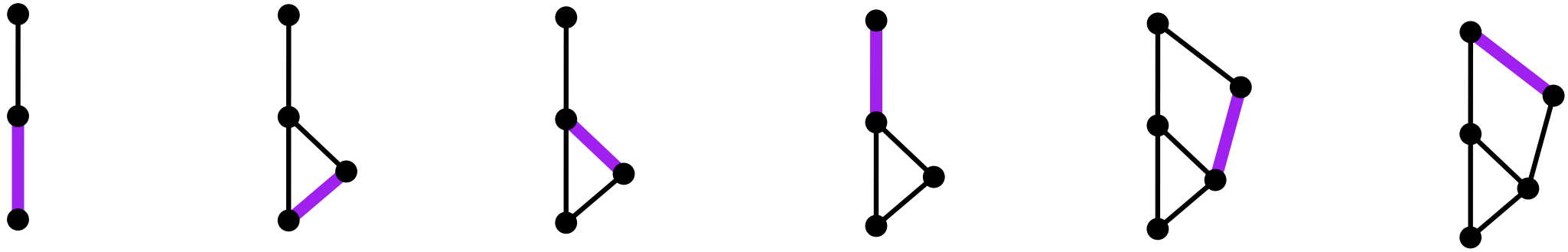


The KMSW bijection

Example: build orientation associated to

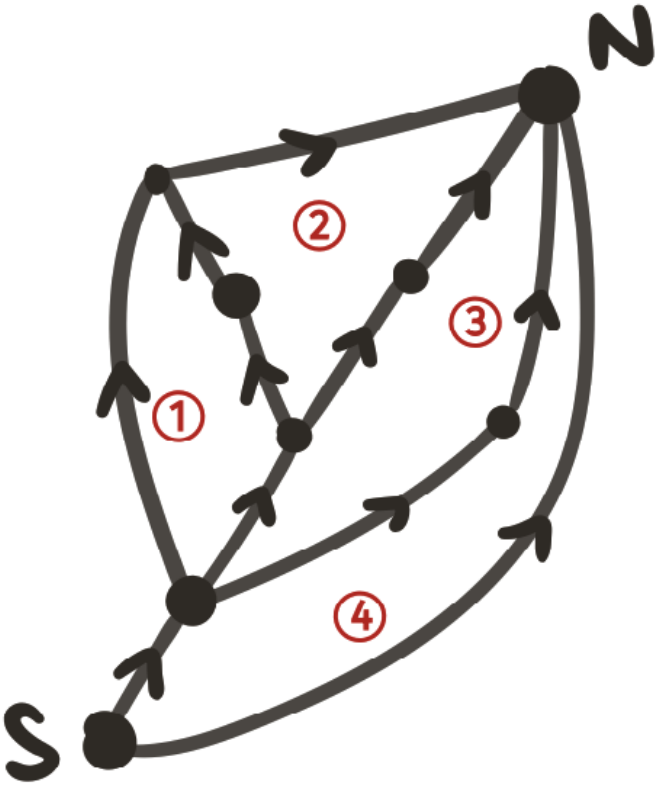


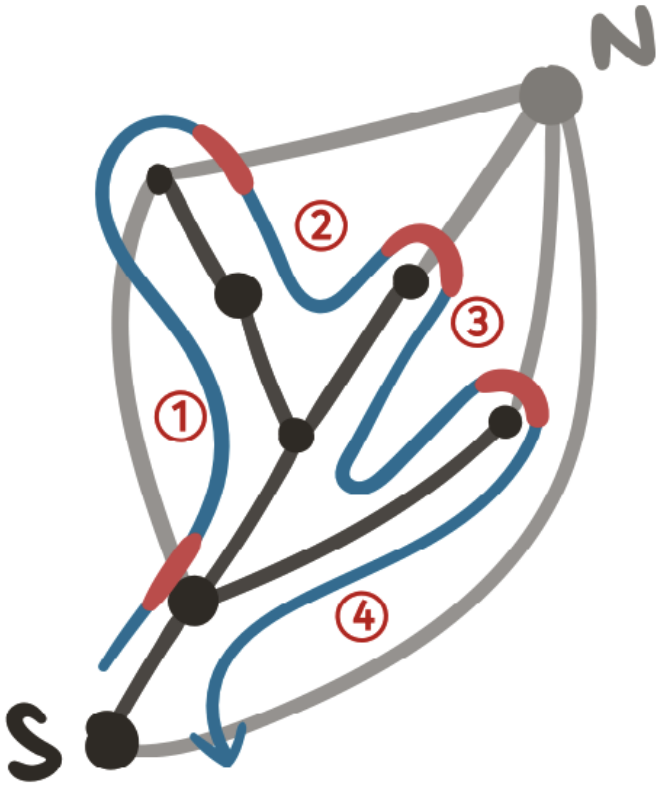


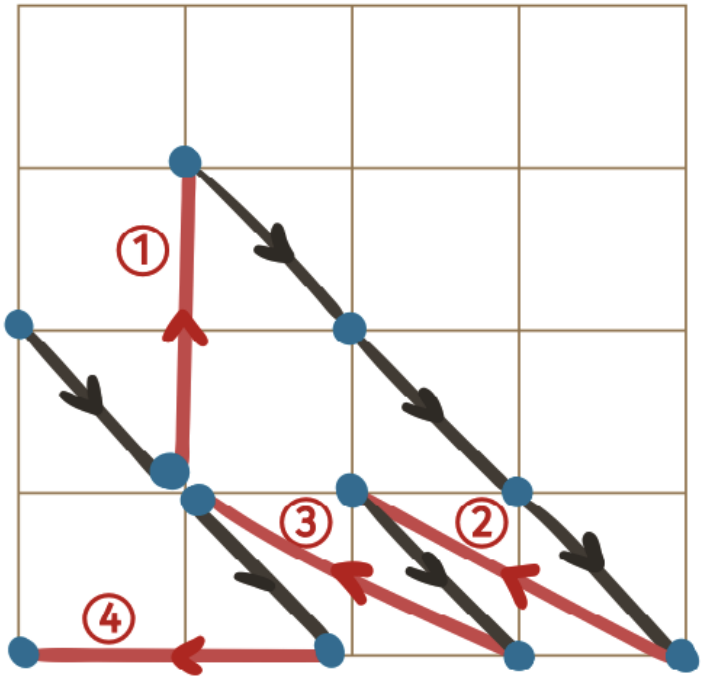


The KMSW bijection

From bipolar orientation to tandem walk

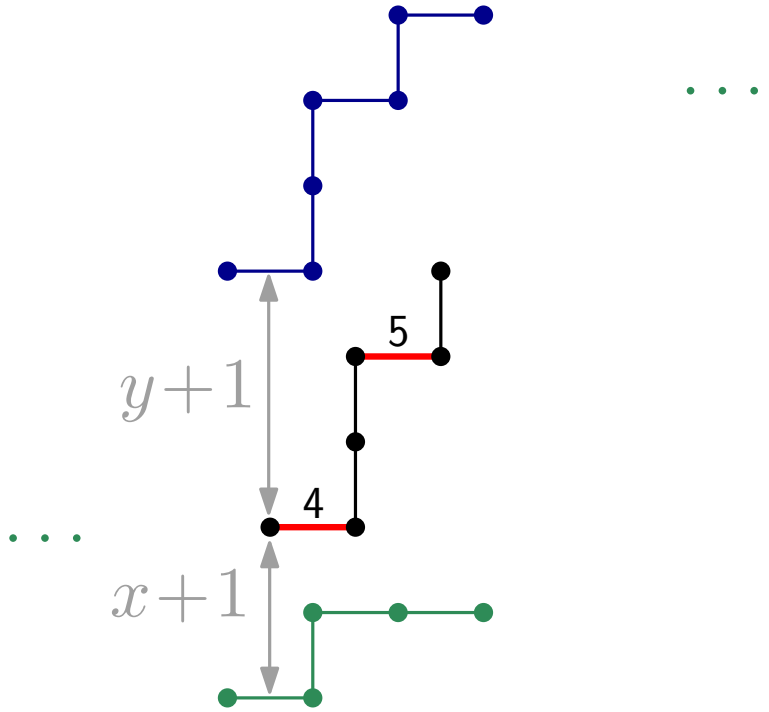




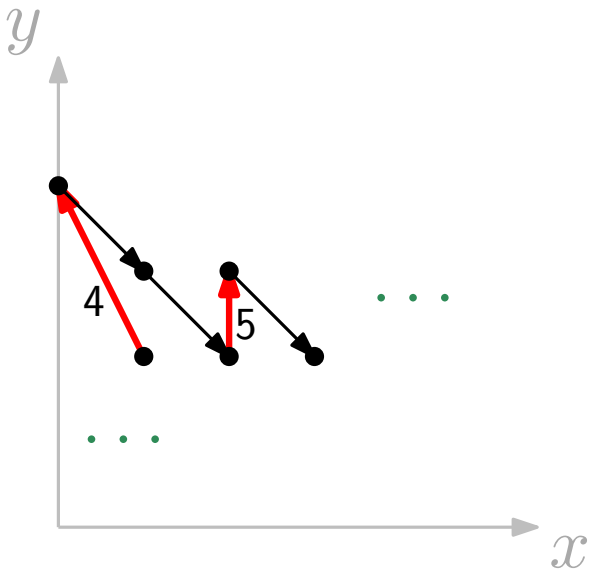


Link with non-intersecting triples of walks

[Bousquet-Mélou,F,Raschel'20]

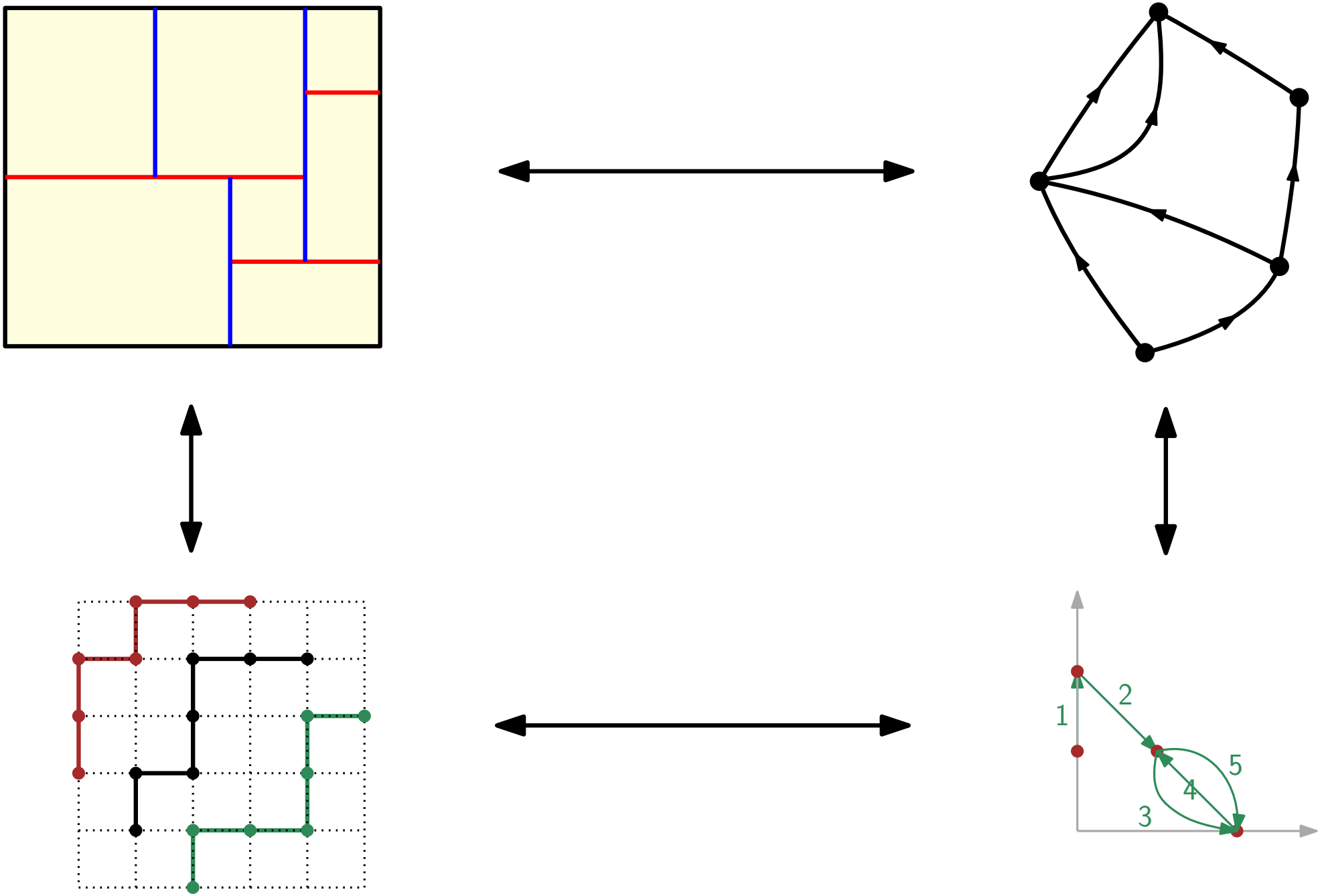


non-intersecting triple

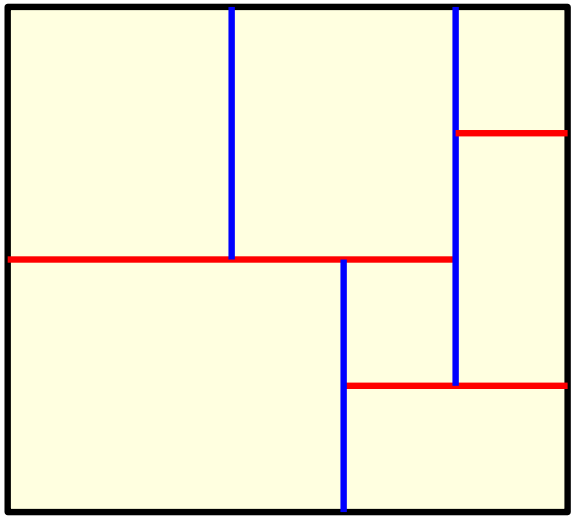


tandem walk

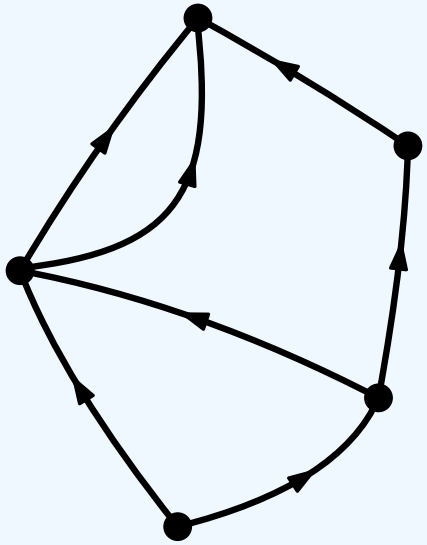
Summary of bijections so far



Summary of bijections so far

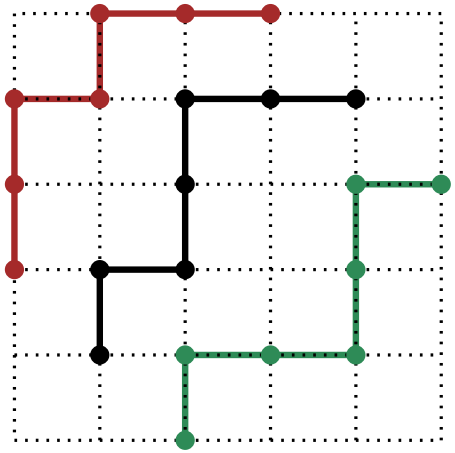




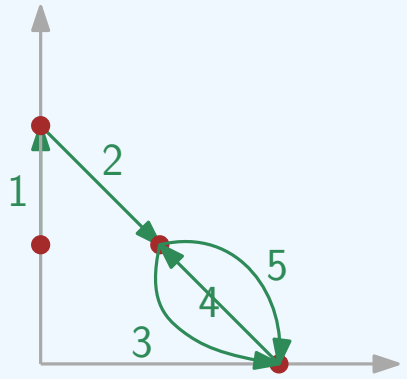




black box
to treat
other models

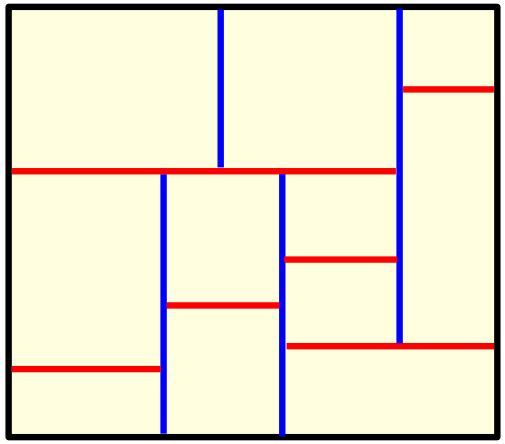


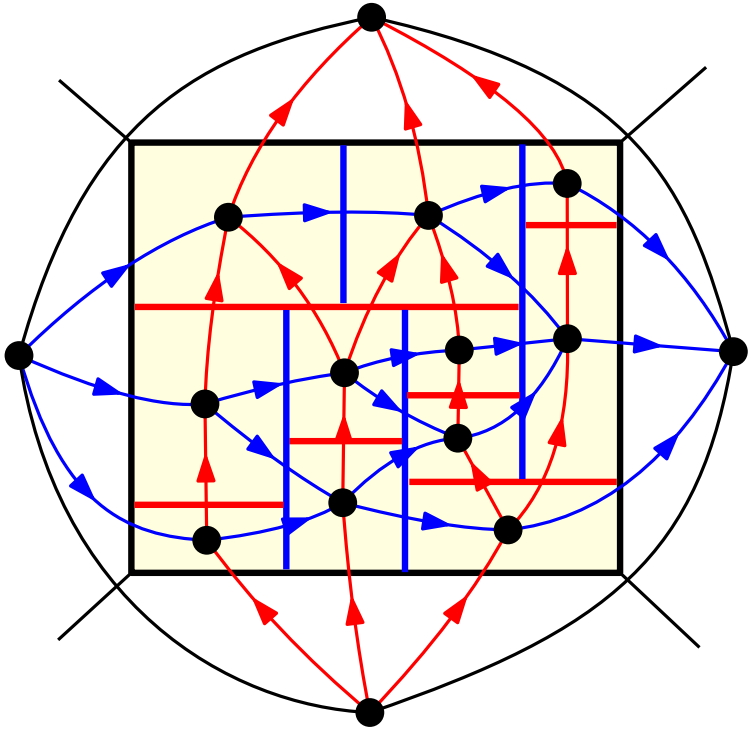


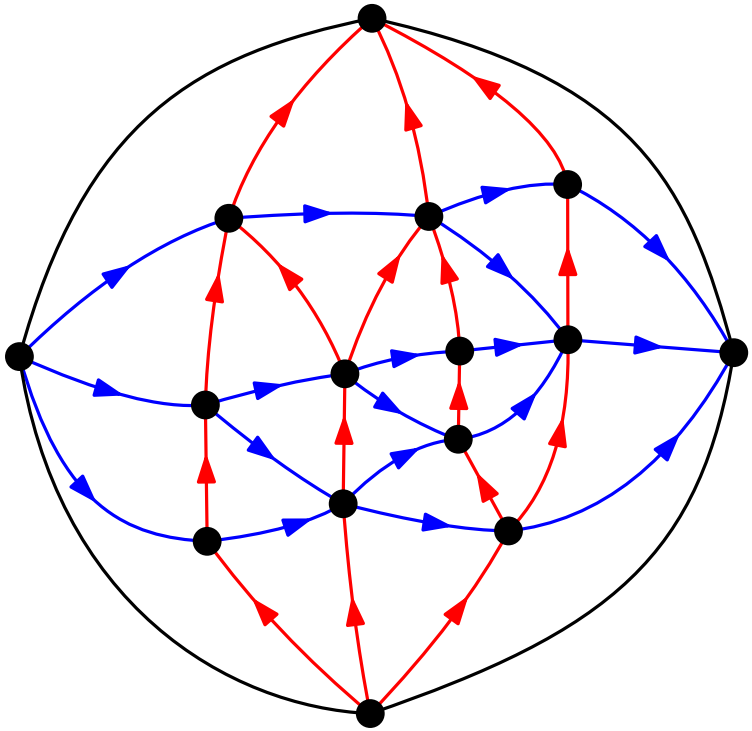


Strong rectangulations

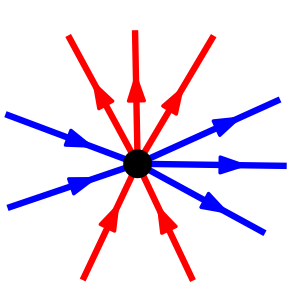
[He'93]

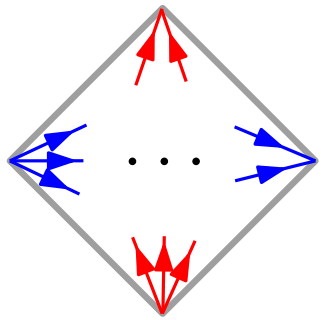






Pair of transversal
plane bipolar orientations

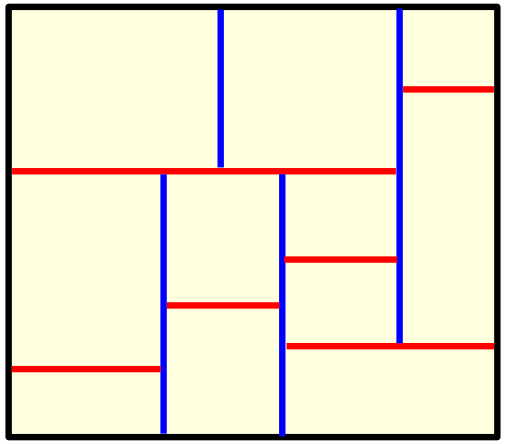


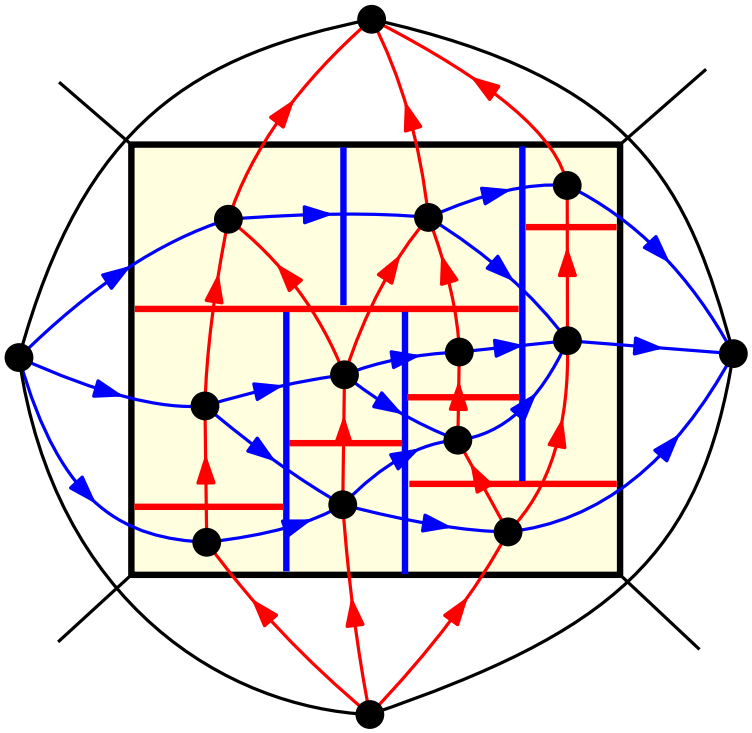


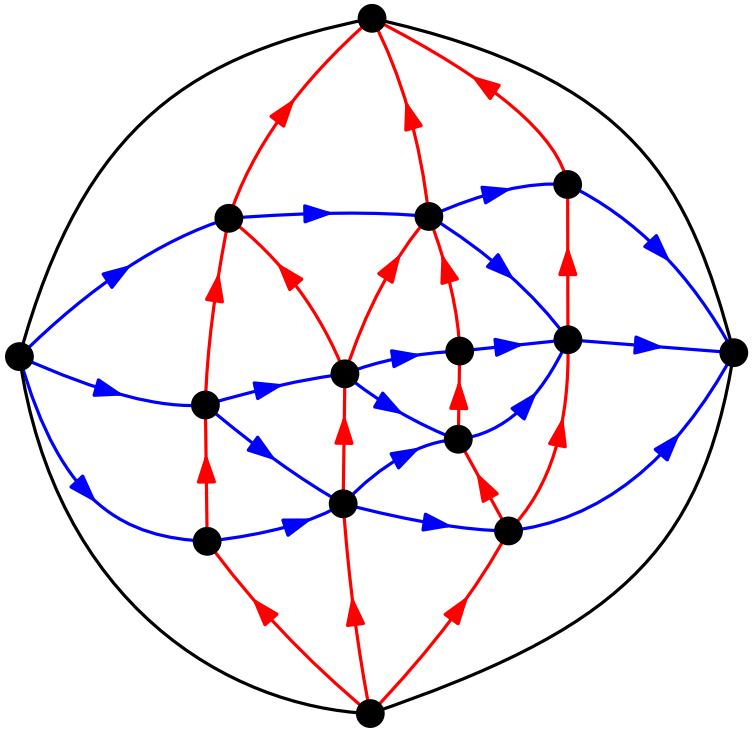
Local conditions

Strong rectangulations

[He'93]



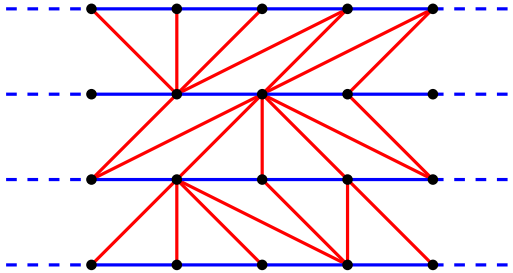




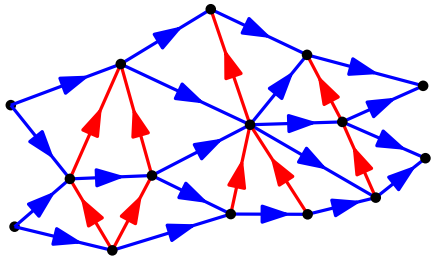
Pair of transversal
plane bipolar orientations

cf locally causal dynamical triangulations

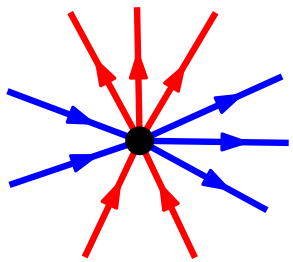
[Loll, Ruijl'15]

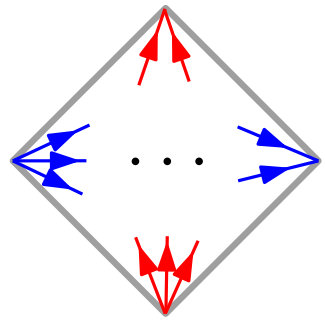


causal



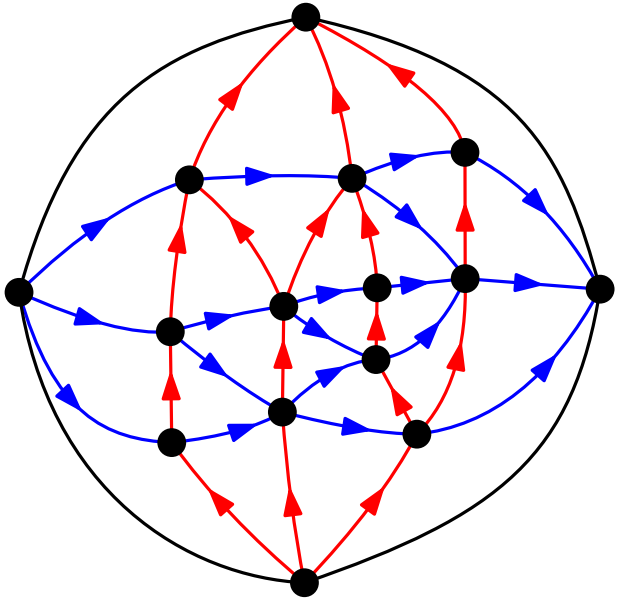
locally causal





Local conditions

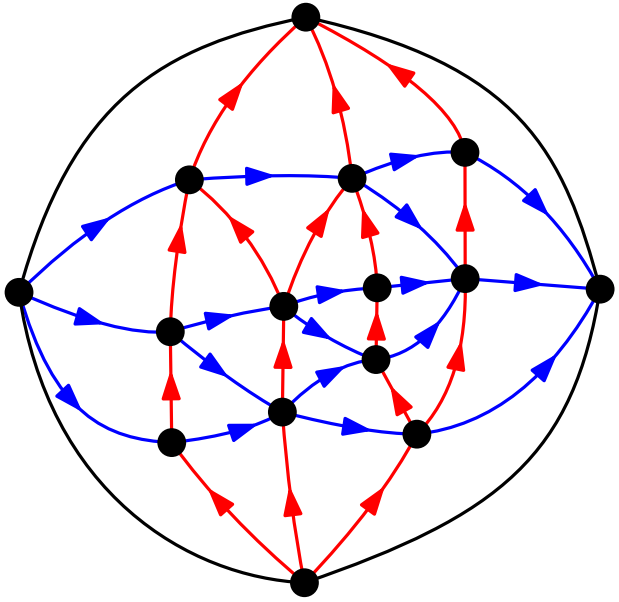
Encoding by (weighted) tandem walks



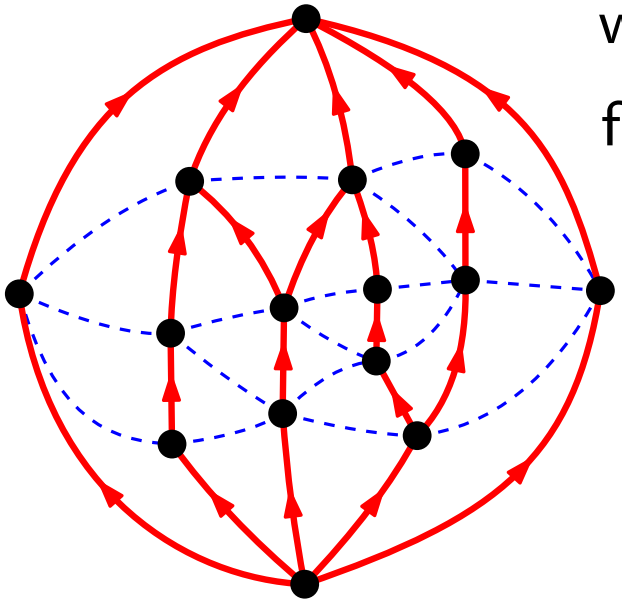
Transversal structure
$n + 4$ vertices

Encoding by (weighted) tandem walks

[F-Narmanli-Schaeffer'21]



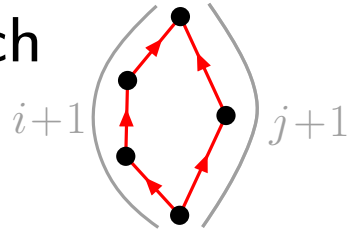
Transversal structure
$n+4$ vertices

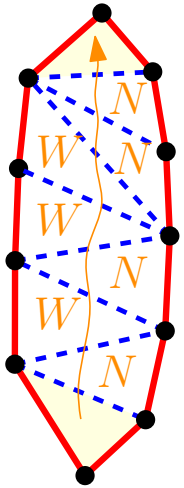


red bipolar poset
$+$ transversal edges

weight $\binom{i+j-2}{i-1}$

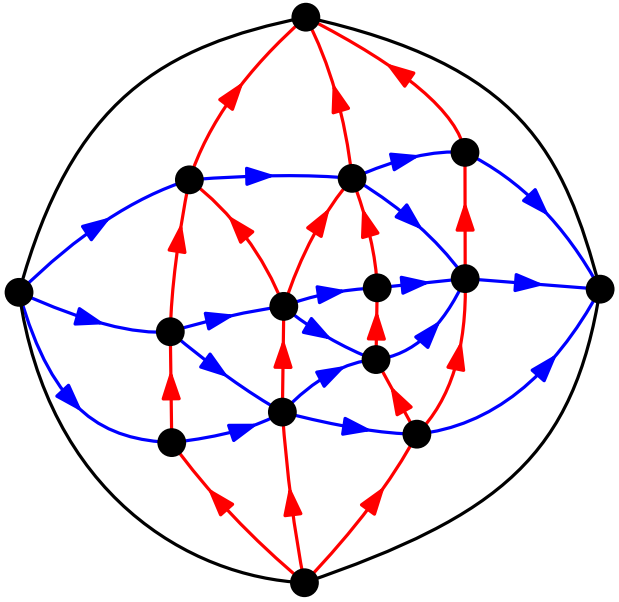
for each



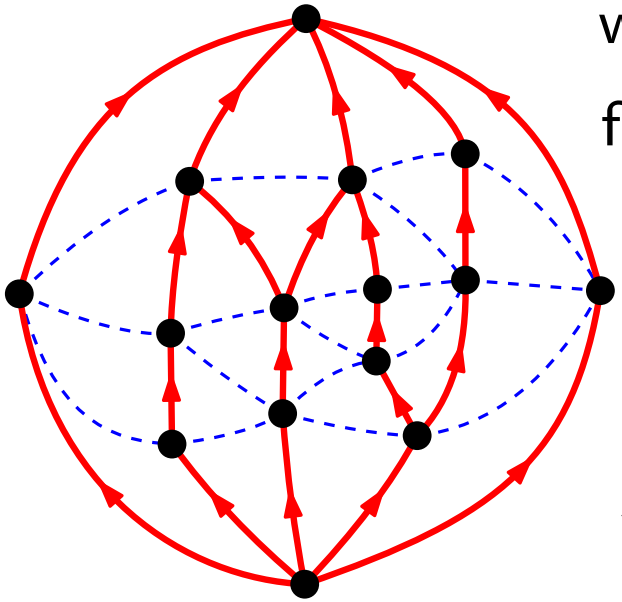


Encoding by (weighted) tandem walks

[F-Narmanli-Schaeffer'21]

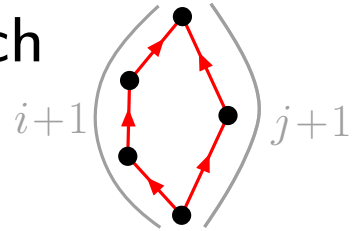


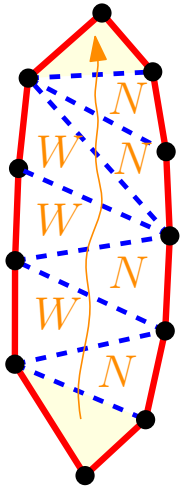
Transversal structure
$n+4$ vertices



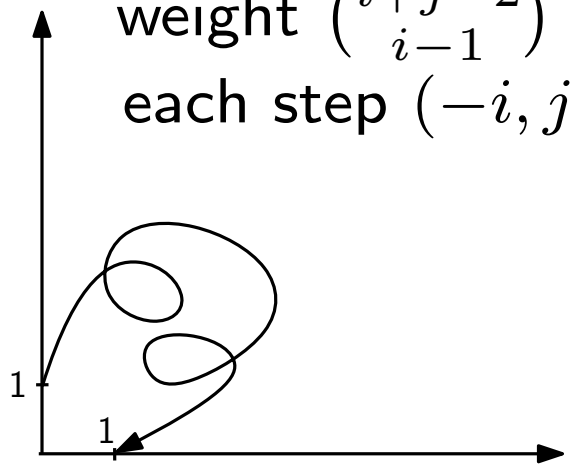
red bipolar poset
+ transversal edges

weight $\binom{i+j-2}{i-1}$
for each





weight $\binom{i+j-2}{i-1}$ for
each step $(-i, j)$

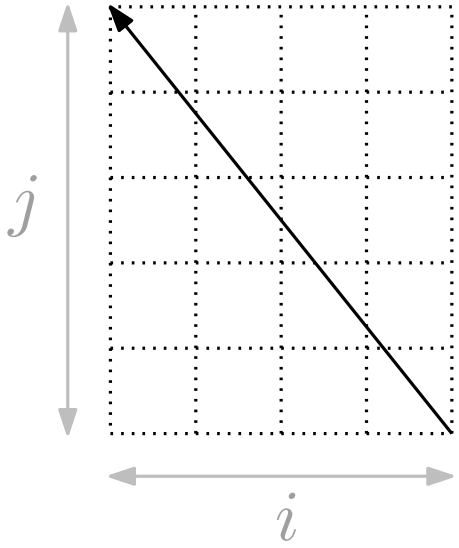


weighted tandem walk
with n SE steps

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

face-step

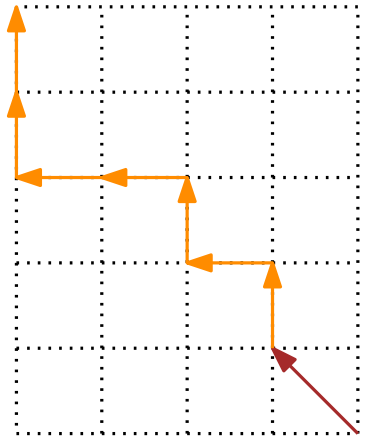


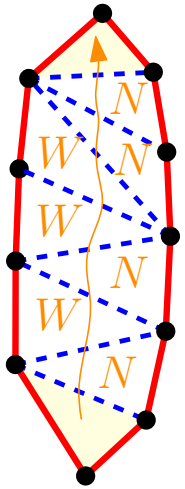
weight

$\binom{i+j-2}{i-1}$

$\Leftrightarrow$

decoration by small-step portion

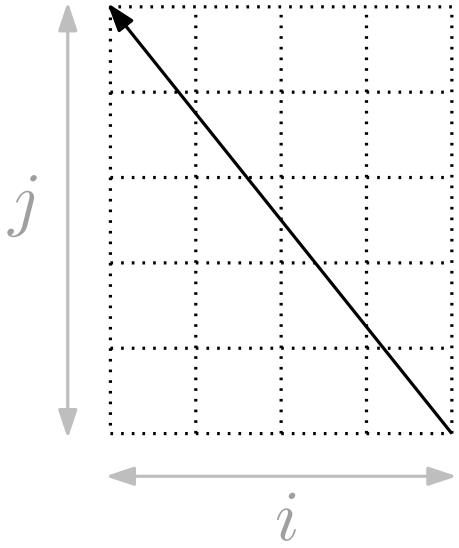




Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

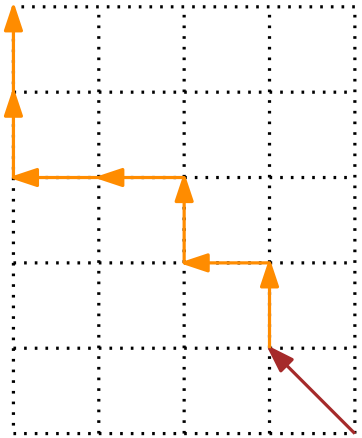
face-step

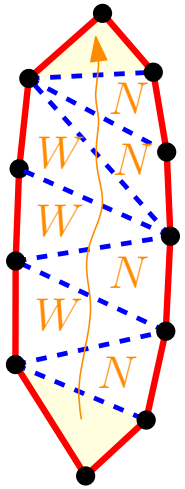


weight $\binom{i+j-2}{i-1}$

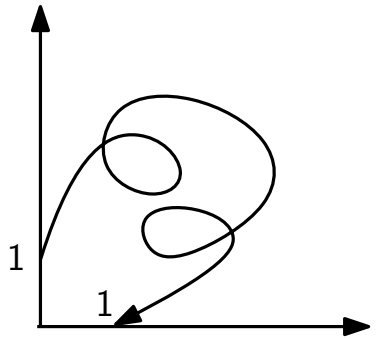
$\Leftrightarrow$

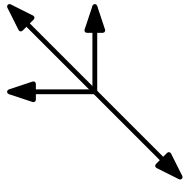
decoration by small-step portion





$s_n = \#$ walks



on step-set 

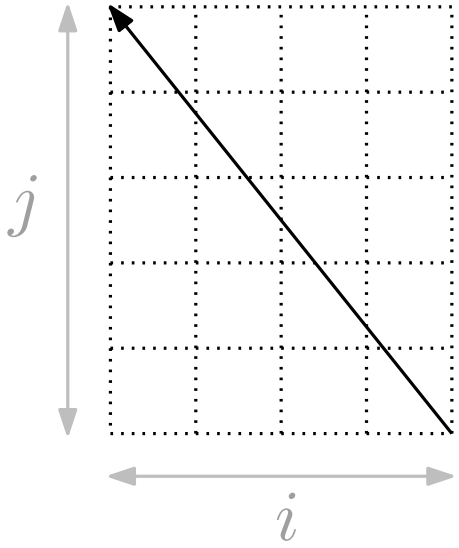
$n-2$ steps 

with no patterns 

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

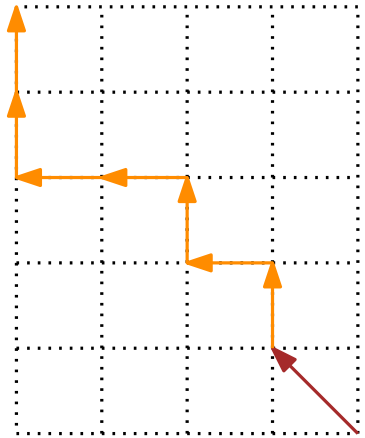
face-step

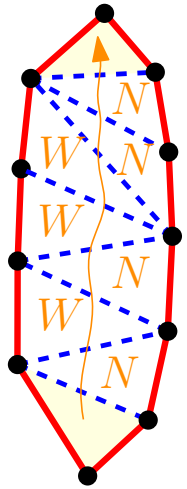


weight $\binom{i+j-2}{i-1}$

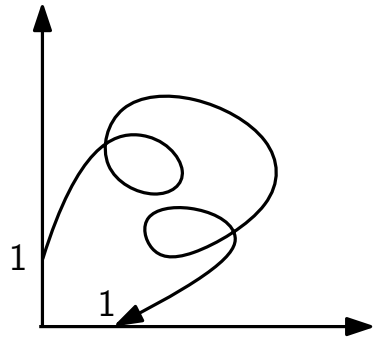
$\Leftrightarrow$

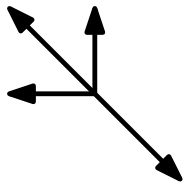
decoration by small-step portion





$s_n = \#$ walks



on step-set 

$n-2$ steps 

with no patterns 

$\Rightarrow$ explicit recurrence

$$\sum_n s_n t^n = t + 2t^2 + 6t^3 + 24t^4 + 116t^5 + 642t^6 + 3938t^7 + \cdots$$

other recurrence (& small step walks) [Inoue, Takahashi, Fujimaki'09]

Asymptotic enumeration

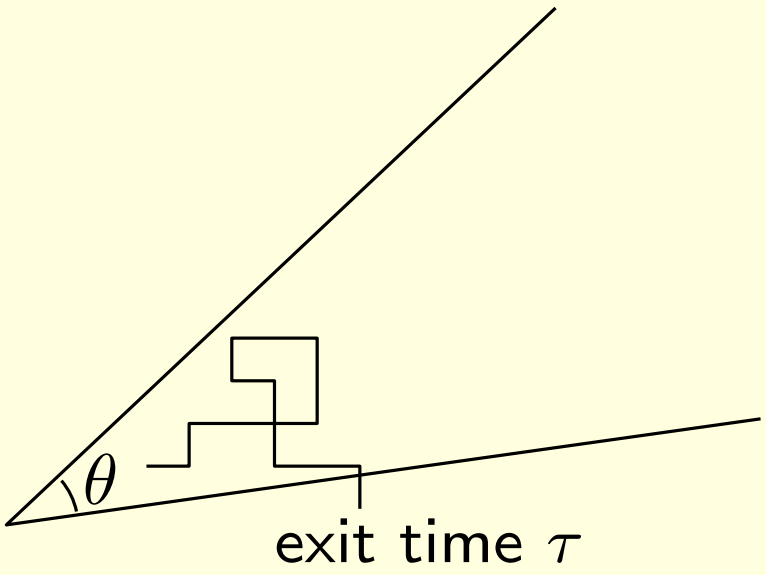
[F-Narmanli-Schaeffer'21]

relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

Each of the counting sequences w_n, s_n has asymptotics of the form

$$c\ \gamma^n\ n^{-\alpha}$$

α : $1 + \frac{\pi}{\theta}$



$$\mathbb{P}(\tau > n) \sim c\ n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \ \& \ \text{excursion}) \sim c'\ n^{-1-\frac{\pi}{\theta}}$$

	weak	strong
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$

Asymptotic enumeration

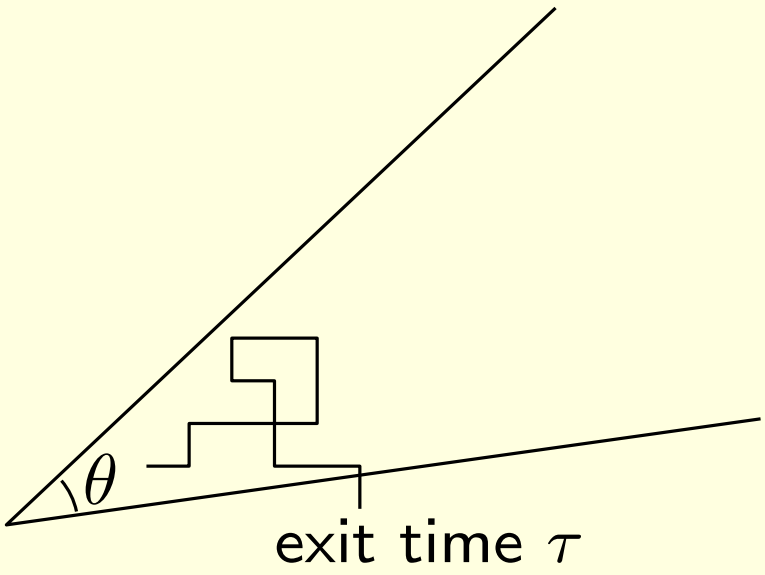
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not D-finite

Asymptotic enumeration

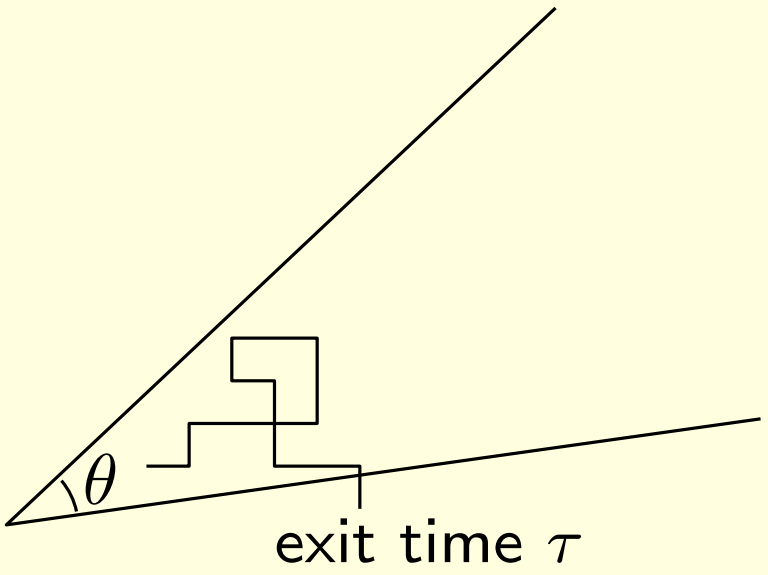
[F-Narmanli-Schaeffer'21]

relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

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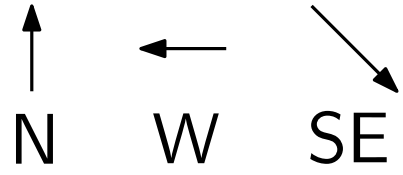
optimal encoding

[Takahashi, Fujimaki, Inoue'09]

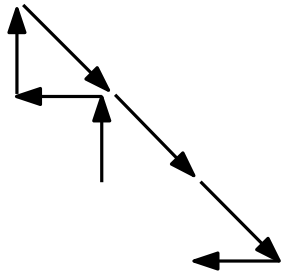
$$s_n \leq \binom{3n}{n} 2^n$$

not D-finite

Illustration on tandem walks with small steps

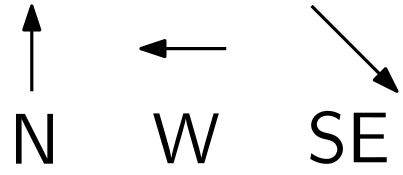
Step-set 

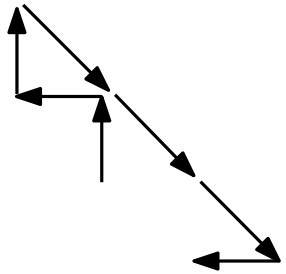
(triangulated bipolar orientations)

Random walk 

$$\mathbb{P}(\text{each step}) = \tfrac{1}{3}$$

Illustration on tandem walks with small steps

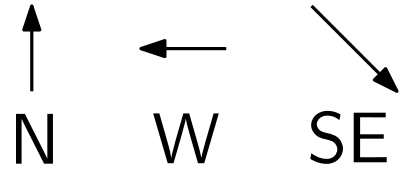
Step-set  (triangulated bipolar orientations)

Random walk 

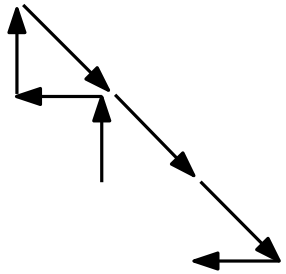
$$\mathbb{P}(\text{each step}) = \tfrac{1}{3}$$

$$\mathrm{Cov} = \begin{pmatrix} \mathbb{E}(X^2) & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}(Y^2) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Illustration on tandem walks with small steps

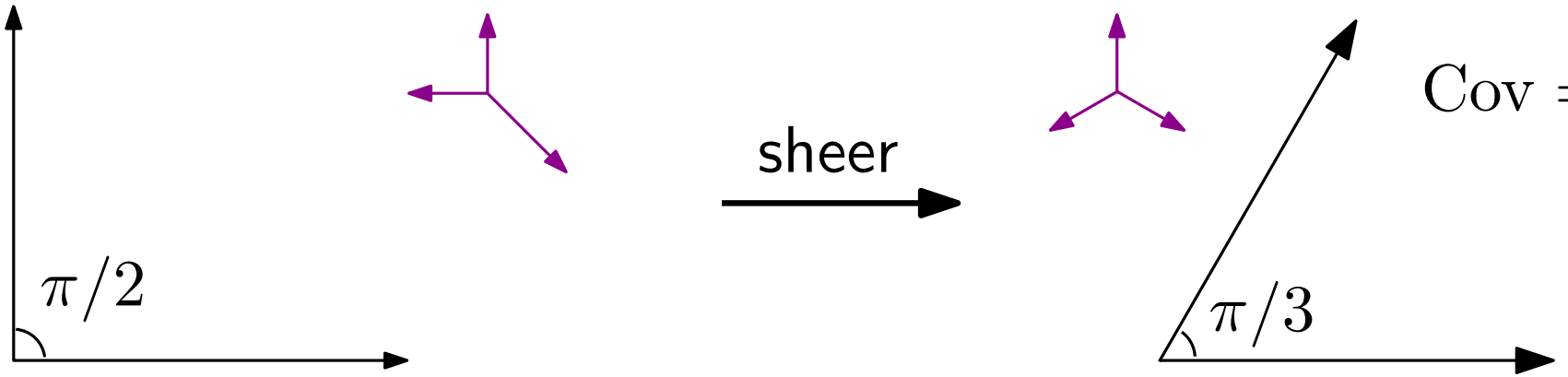
Step-set 

(triangulated bipolar orientations)

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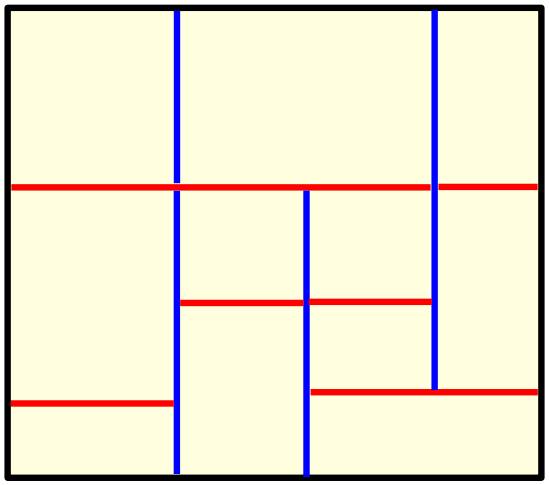


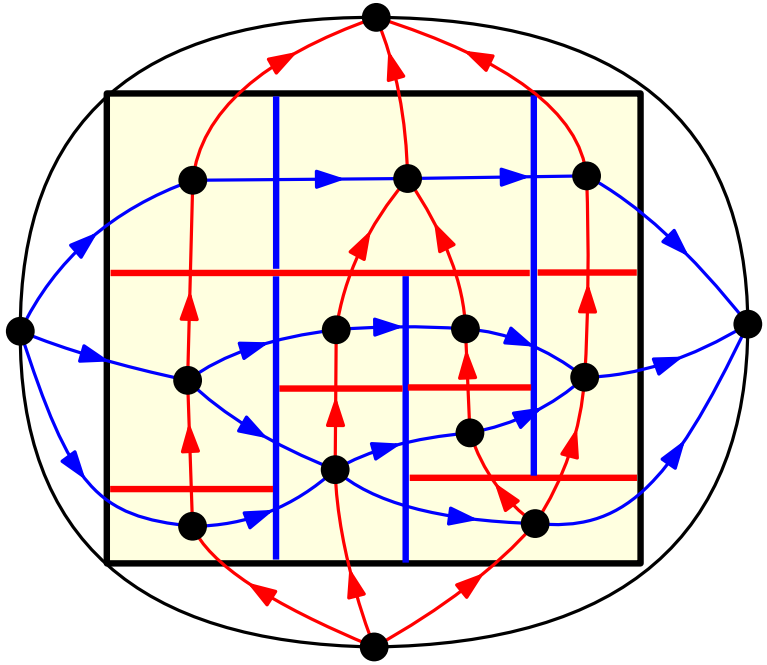
$$\text{Cov} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

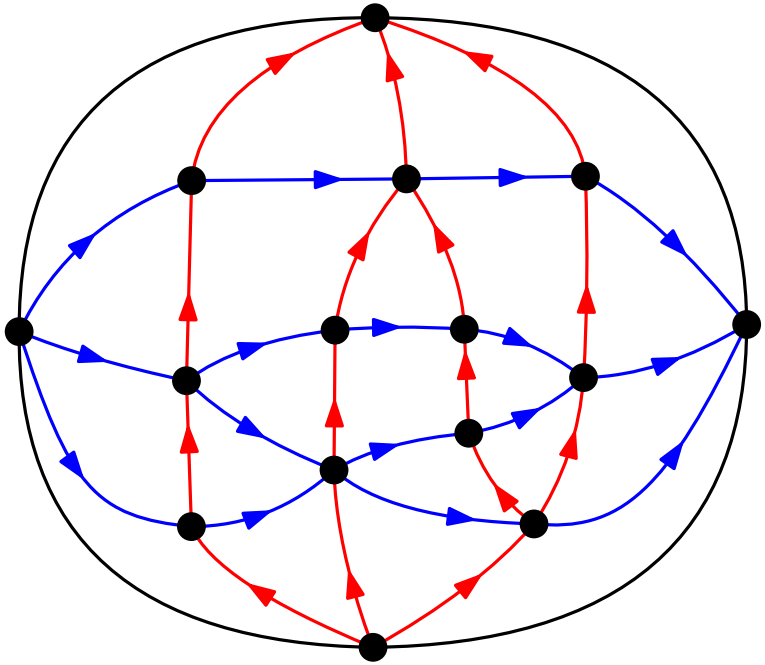
$\Rightarrow$ # quadrant excursions length $3n \sim c \cdot 27^n n^{-4}$

($\alpha = 4$ universal for plane bipolar orientations)

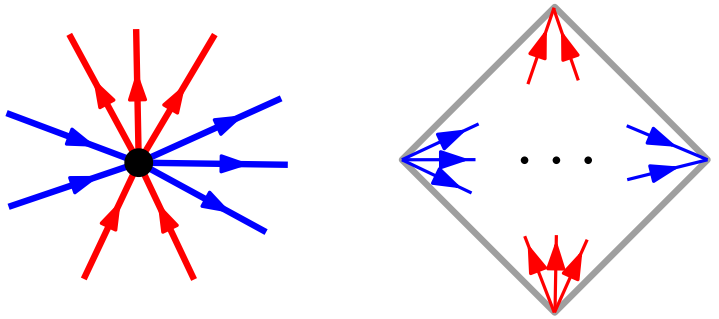
Non-generic rectangulations





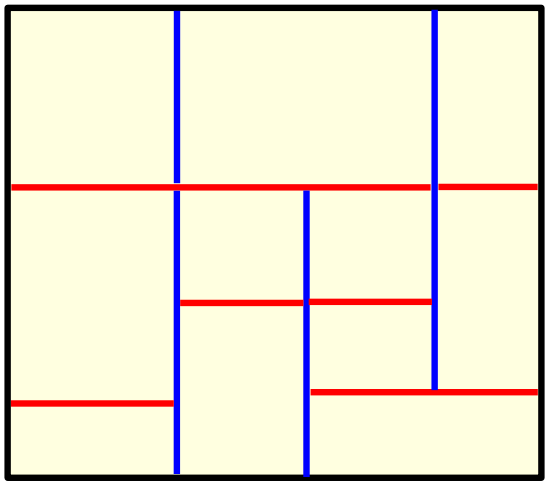


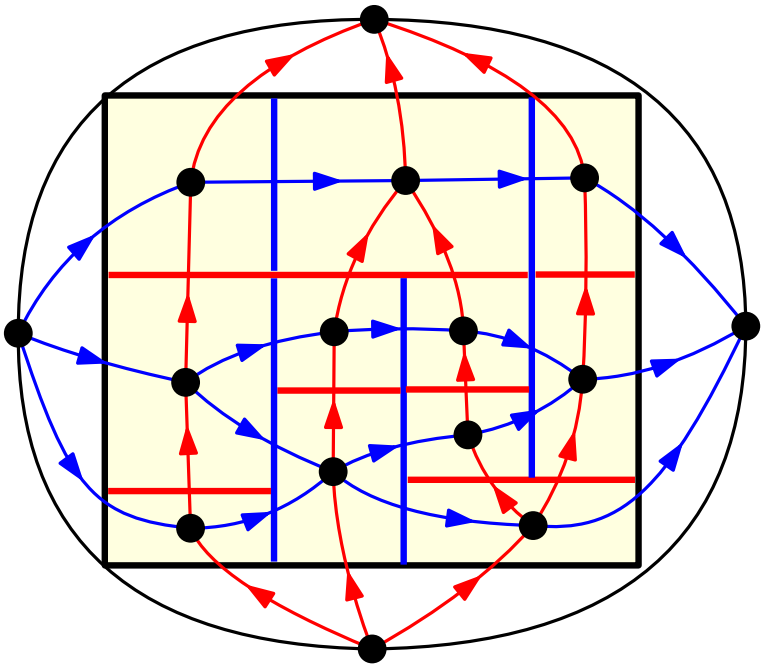
Pair of transversal
plane bipolar orientations

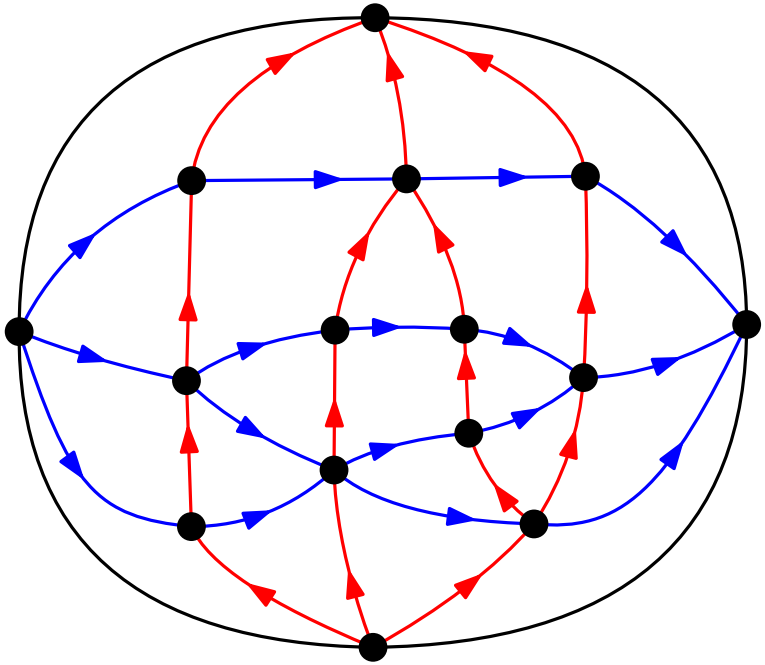


Local conditions

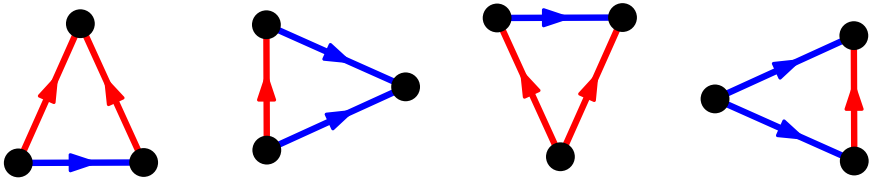
Non-generic rectangulations





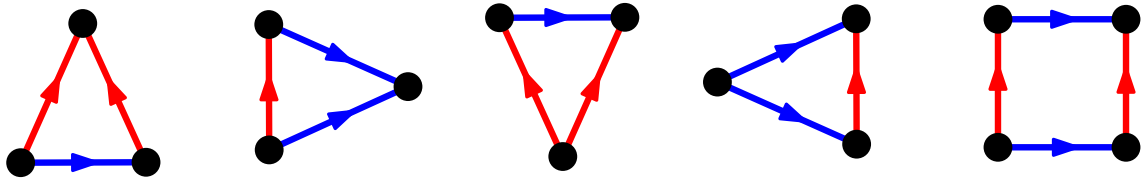


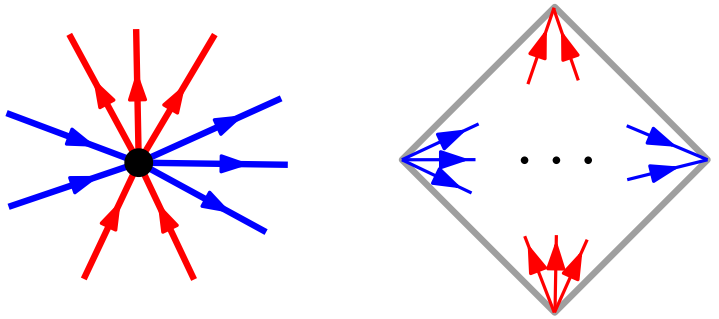
Generic case: inner faces of 4 types



Pair of transversal
plane bipolar orientations

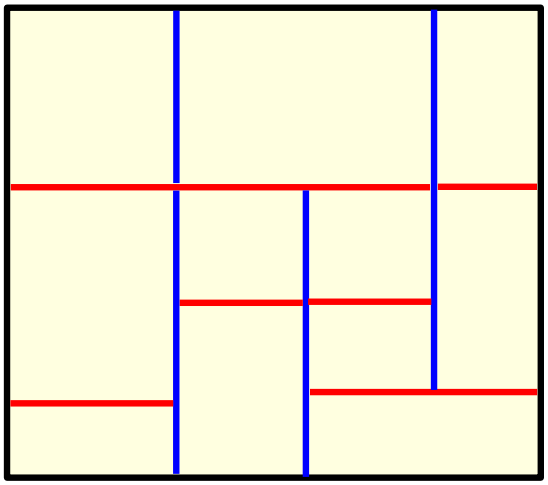
Non-generic case: inner faces of 5 types

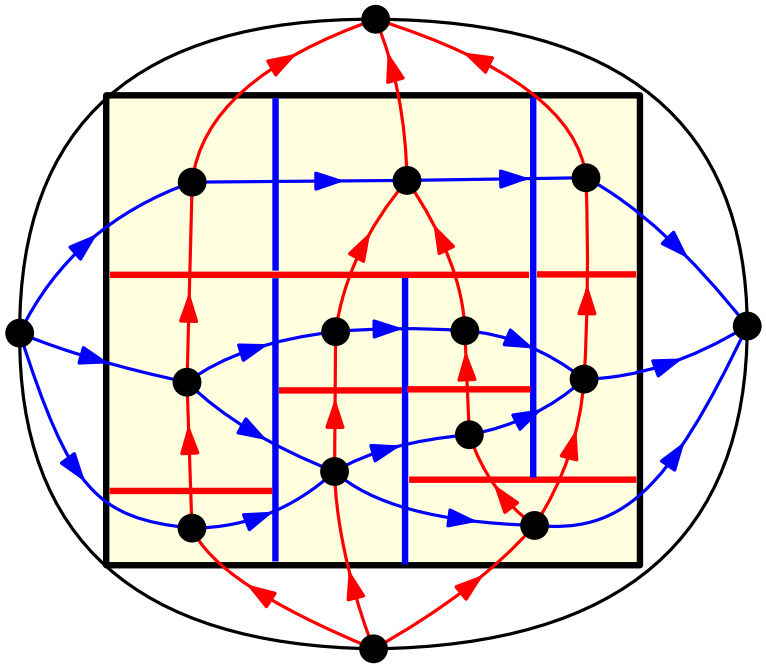


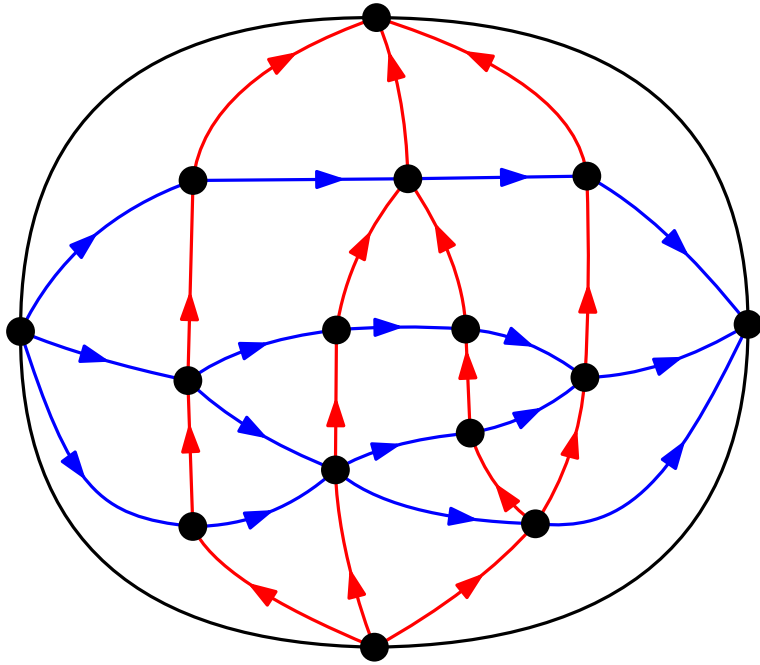


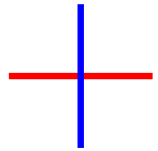
Local conditions

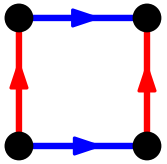
Non-generic rectangulations with weighted quadrangles



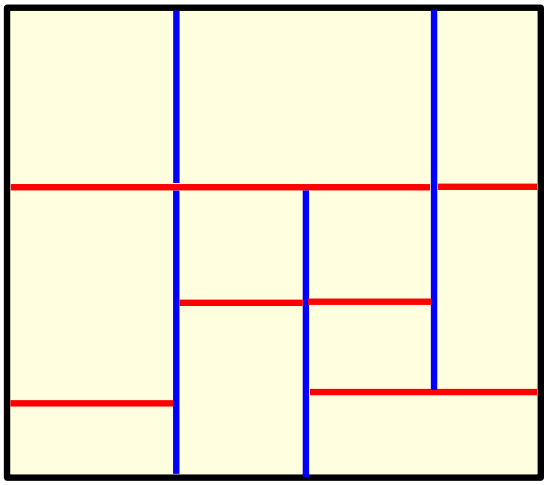


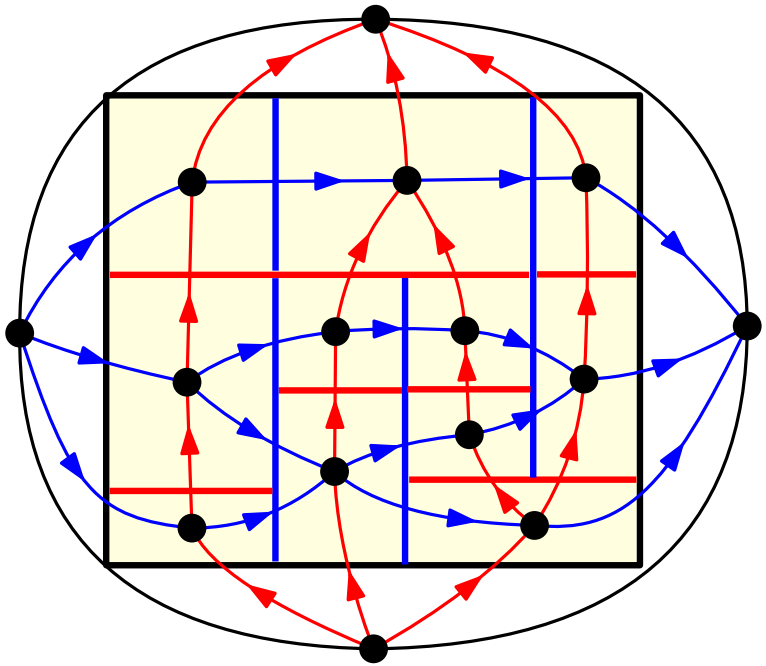


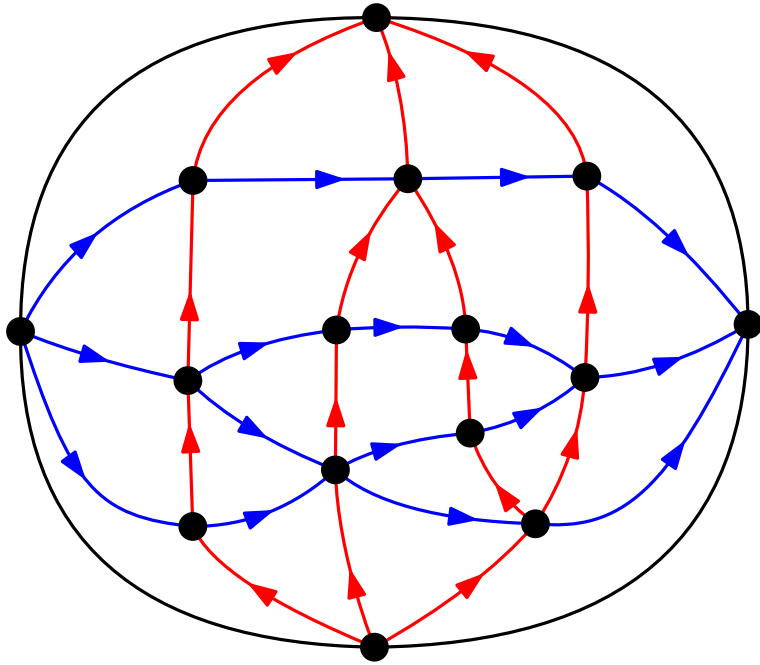
Let $s_n(v) = \#$ rectangulations with n regions, weight v per 

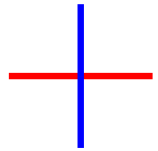
$= \#$ transversal structures with $n+4$ vertices, weight v per 

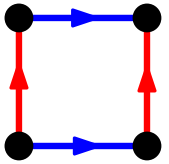
Non-generic rectangulations with weighted quadrangles





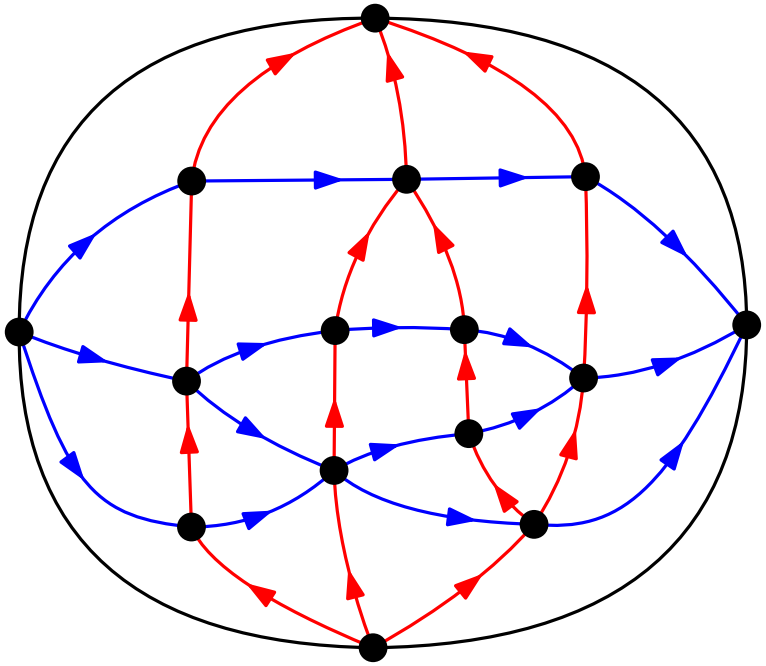


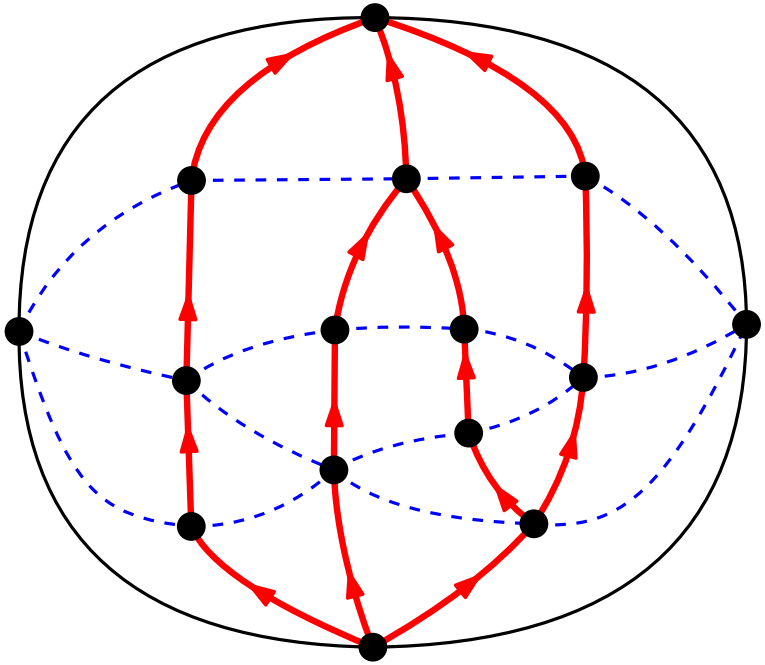
Let $s_n(v) = \#$ rectangulations with n regions, weight v per 

$= \#$ transversal structures with $n+4$ vertices, weight v per 

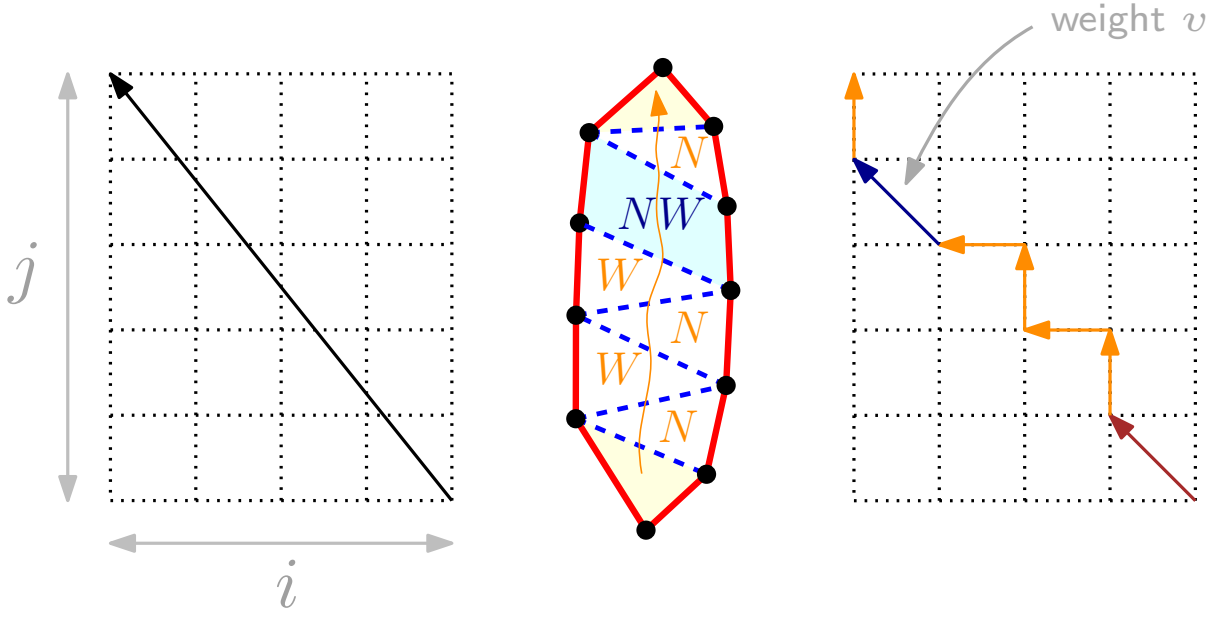
$$\sum_n s_n(v)t^n = t + 2t^2 + 6t^3 + (24+v)t^4 + (116+12v)t^5$$
$$+(642+114v+2v^2)t^6 + (3938+1028v+48v^2)t^7 + \cdots$$

Encoding by tandem walks





small-step portion for a face



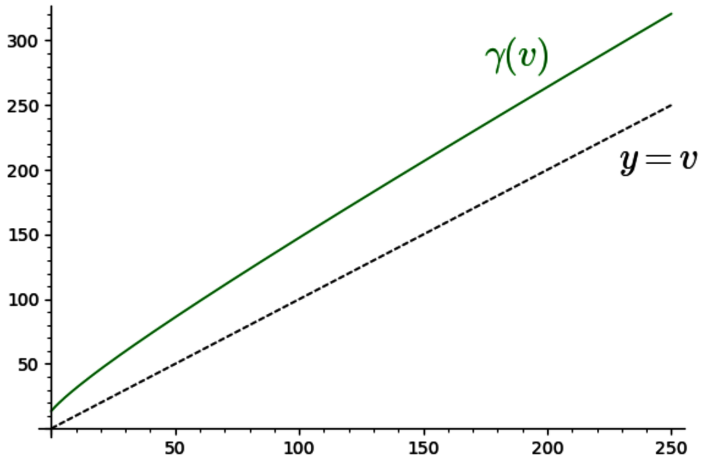
weight on face-step $(-i, j)$: $w(i,j) = \sum_k v^k \binom{i+j-2-k}{k} \binom{i+j-2-2k}{i-1-k}$

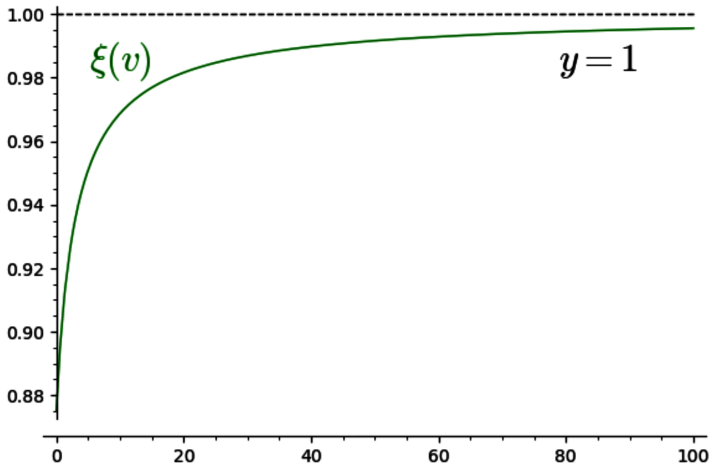
Asymptotic behaviour

[F-Narmanli-Schaeffer'21]

For $v > 0$, $\quad s_n(v) \sim c(v)\ \gamma(v)^n\ n^{-\alpha(v)}$

$\alpha(v) = 1 + \frac{\pi}{\theta(v)}$





$$\gamma(v) = \frac{1}{2(2+v)}\left(2v^2+18\,v+27+(9+4v)^{3/2}\right)$$

$\gamma(0) = 27/2$

$$\xi(v) = \frac{1}{4(2+v)^2}\left(4v^2+14v+11+\sqrt{9+4v}\right)$$

$\xi(0) = 7/8$

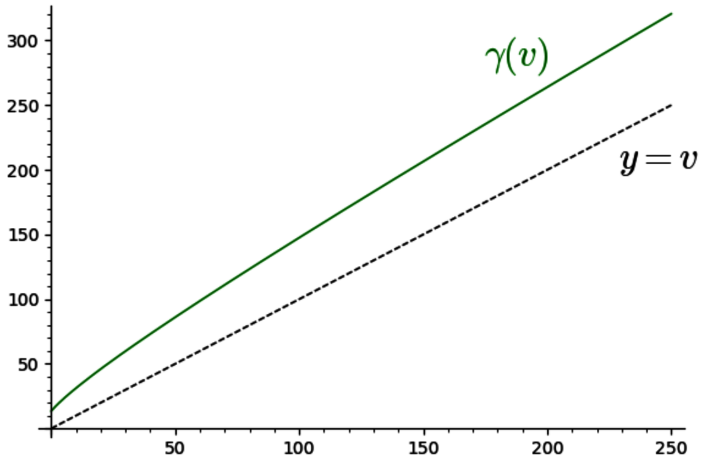
$\xi(v) = \cos(\theta(v))$

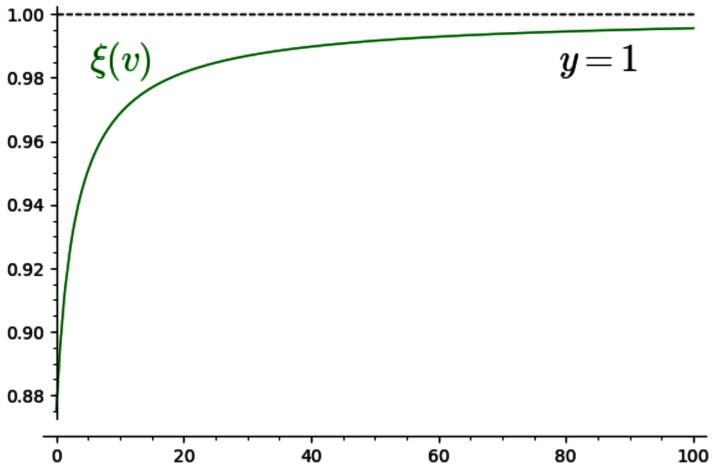
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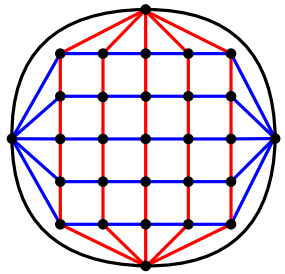
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$\xi(0) = 7/8$

$\cos(\theta(v))$

Regime $v \to \infty$ $\qquad \gamma(v) \sim v, \quad \alpha(v) \to \infty \qquad \sim$ regular grid



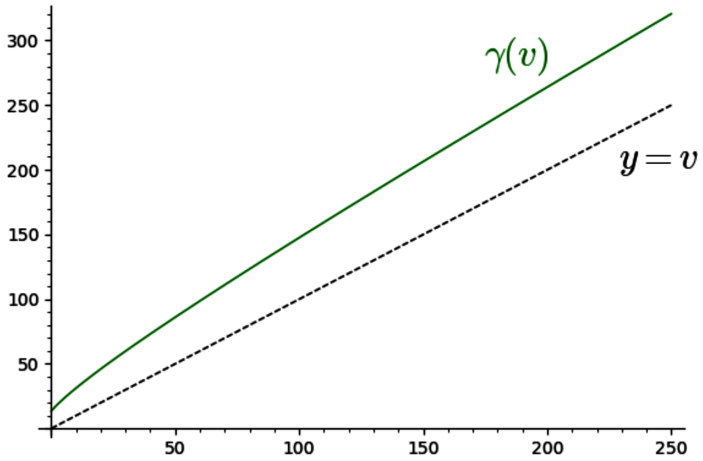
Regime $v = O(1)$ random lattice (universality class evolves with v)

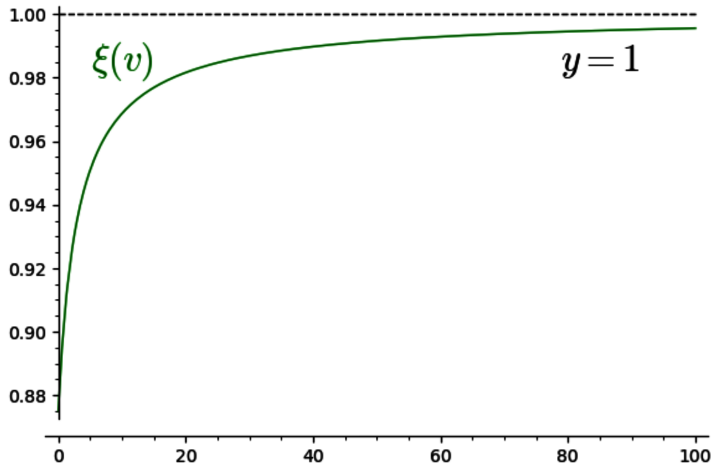
Asymptotic behaviour

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$1 + \frac{\pi}{\theta(v)}$





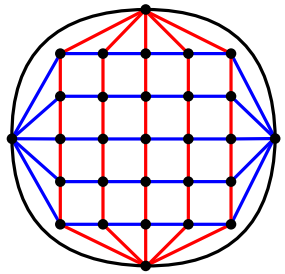
$$\gamma(v) = \frac{1}{2(2+v)}\left(2v^2+18\,v+27+(9+4v)^{3/2}\right)$$

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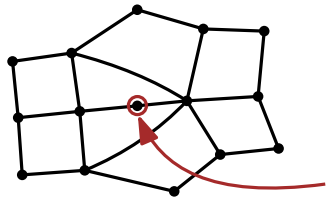
$\cos(\theta(v))$



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Regime $v = O(1)$ random lattice (universality class evolves with v)

cf [Kazakov, Staudacher, Wynter'96]
(other interpolating model)



Eulerian quadrangulations

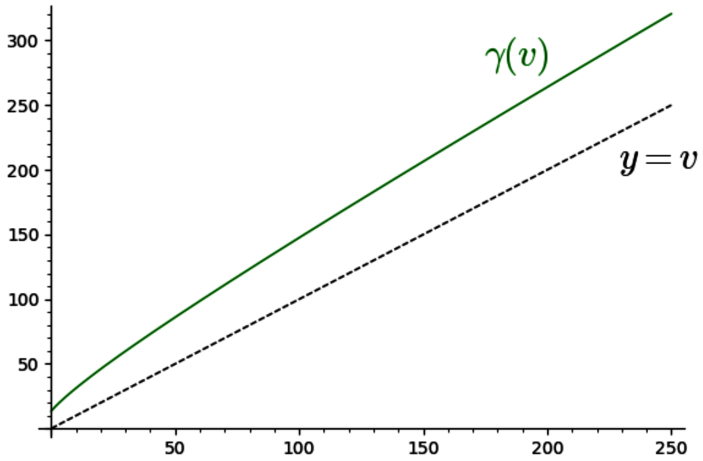
weight v^{-1} per

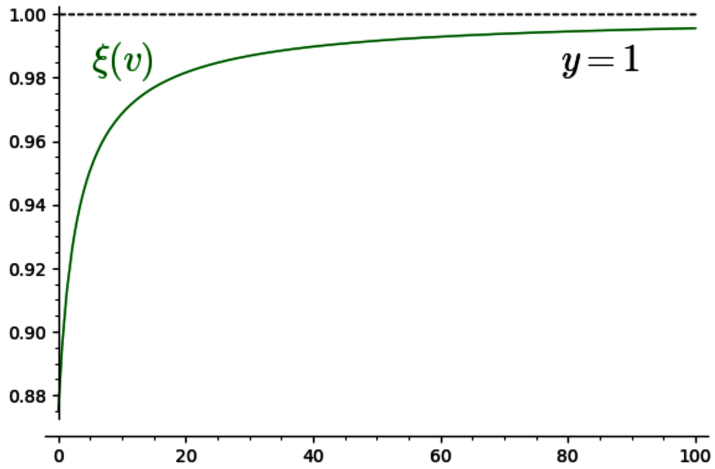
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For $v > 0$, $\quad s_n(v) \sim c(v)\ \gamma(v)^n\ n^{-\alpha(v)}$

$1 + \frac{\pi}{\theta(v)}$





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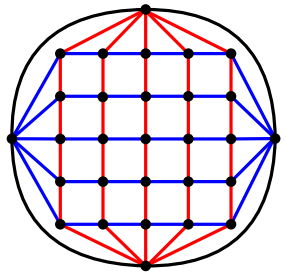
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$\cos(\theta(v))$

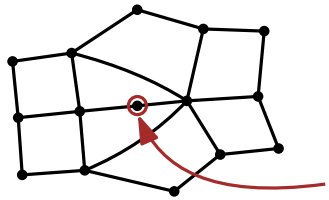
Rk: $\gamma(v), \xi(v) > 0$ for $v \geq -2$

Regime $v \to \infty$ $\qquad \gamma(v) \sim v, \quad \alpha(v) \to \infty$ $\qquad \sim$ regular grid



Regime $v = O(1)$ random lattice (universality class evolves with v)

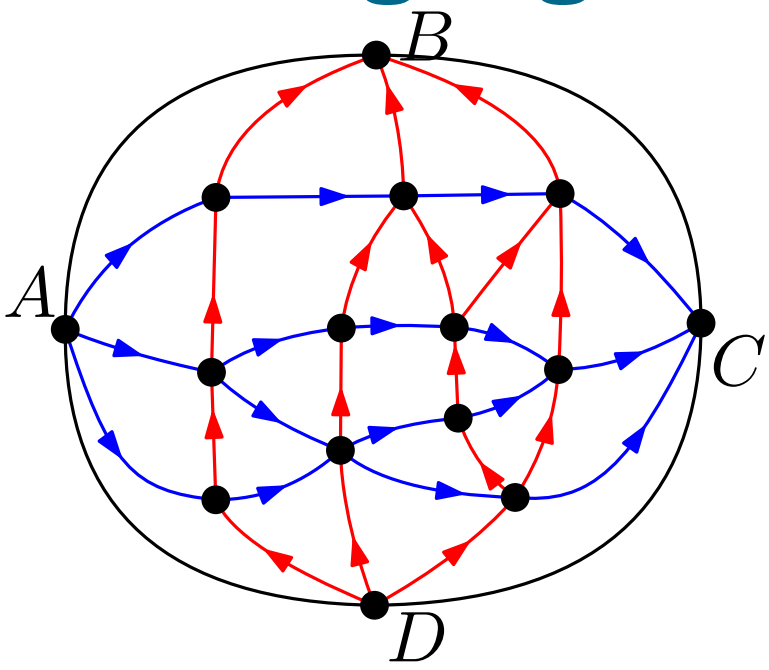
cf [Kazakov, Staudacher, Wynter'96]
(other interpolating model)

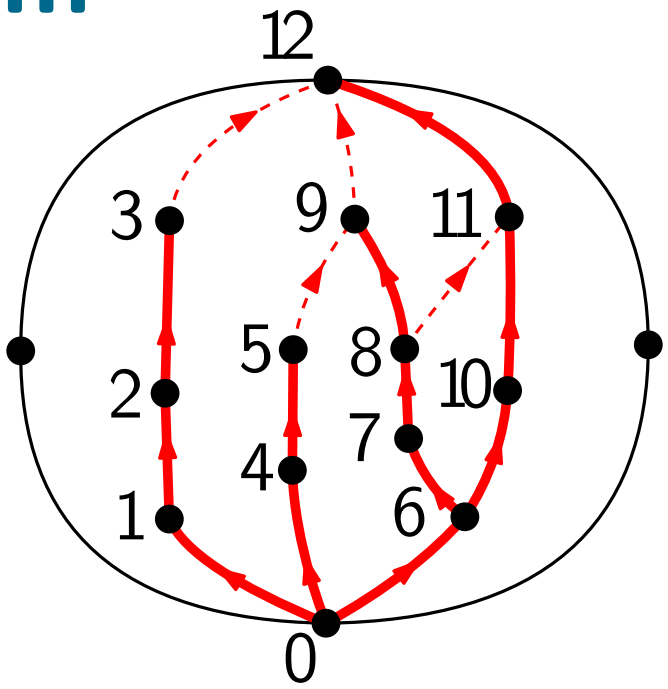


Eulerian quadrangulations

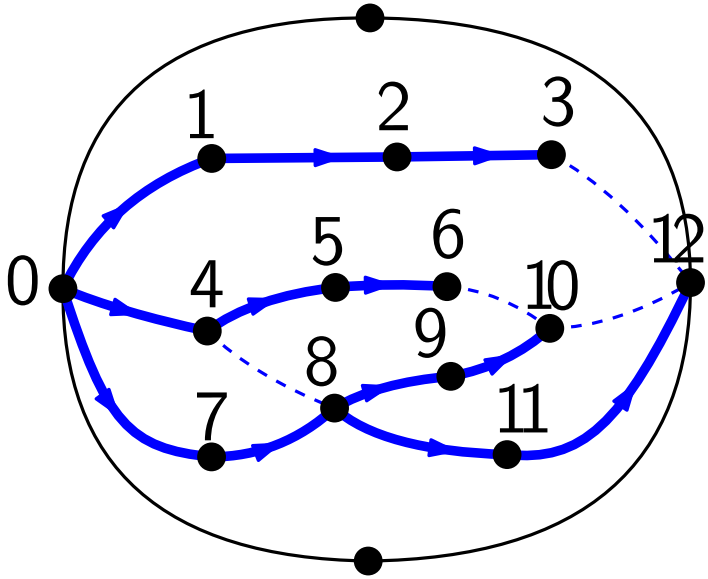
weight v^{-1} per

Drawing algorithm

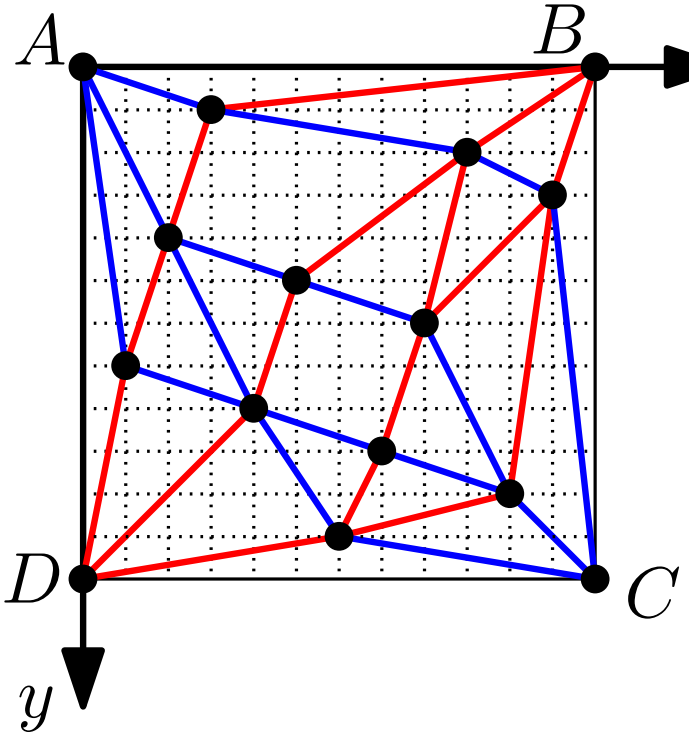


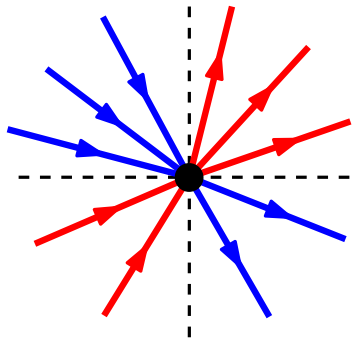


x-coordinates

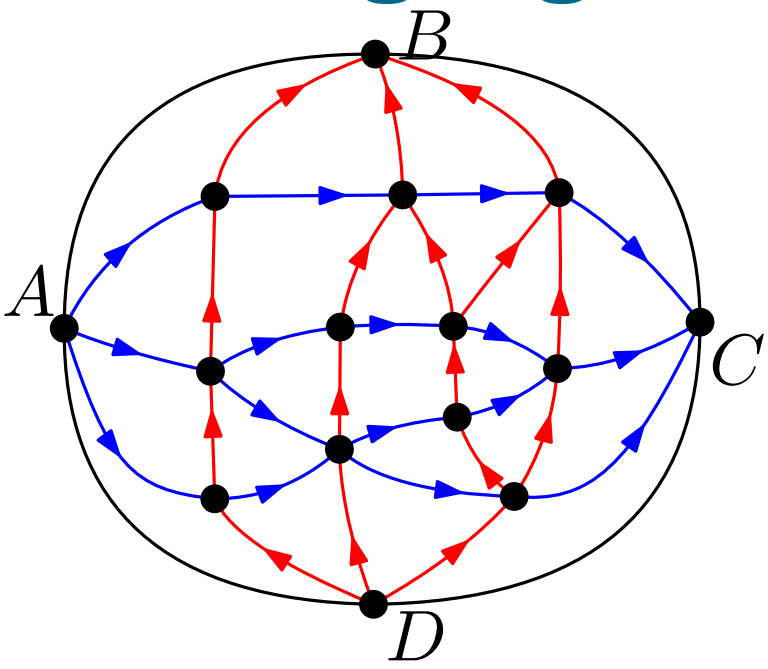


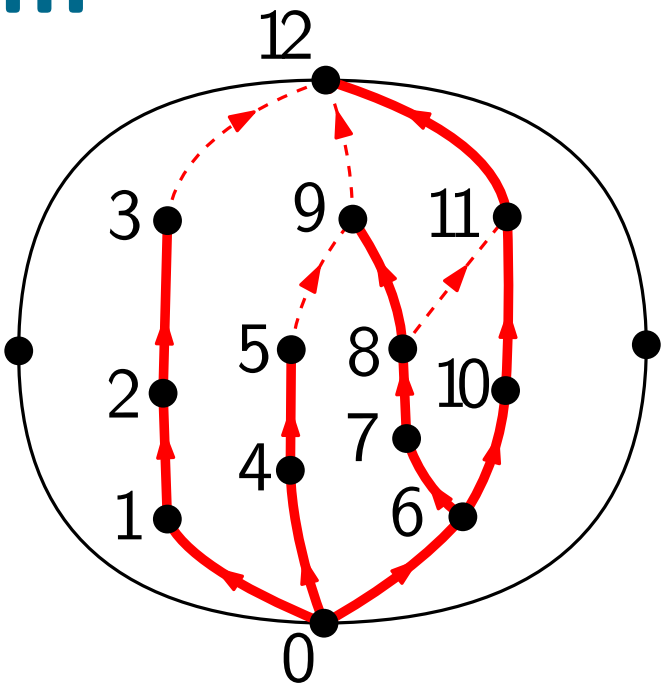
y-coordinates



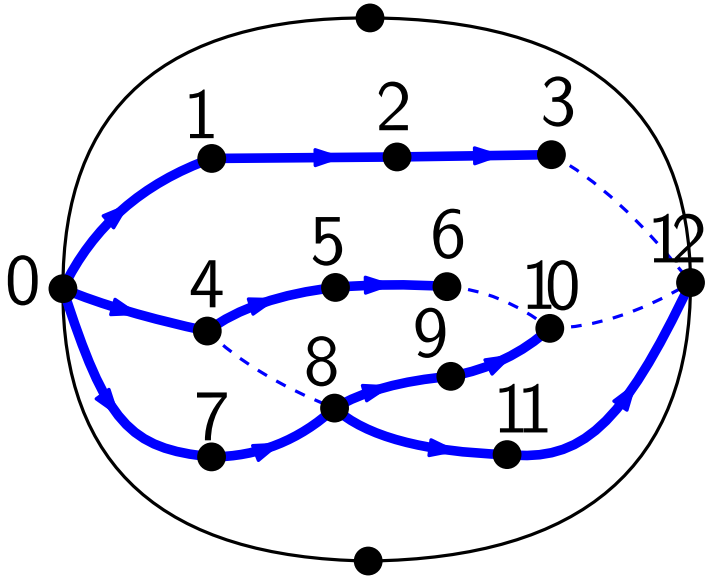


Drawing algorithm

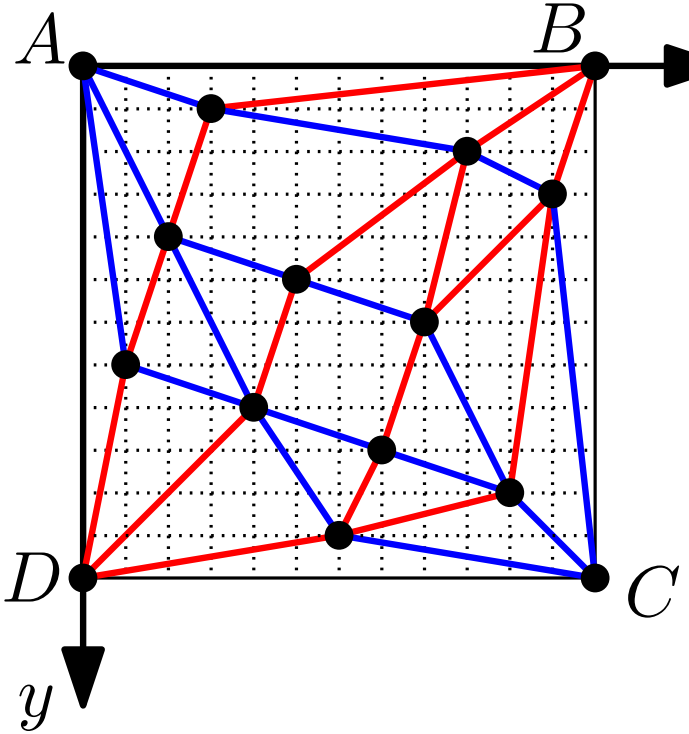


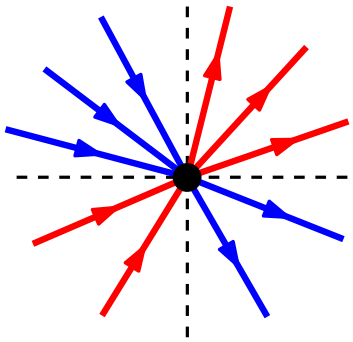


x-coordinates

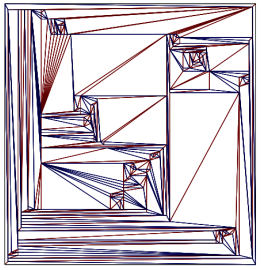


y-coordinates

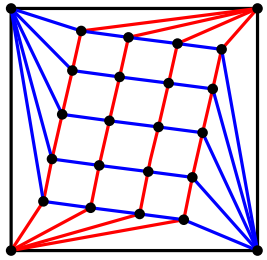




Expected:



for $v = O(1)$



for $v >> 1$

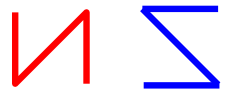
Reduction of a transversal structure

Reduction of a transversal structure (that possibly has quadrangles)

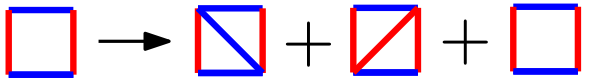
Apply in parallel 

Property: A transversal structure gives same drawing as its reduced form

Expected: Drawing is bijection from reduced transversal structures to semi-Baxter permutations

no 

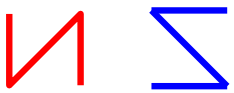
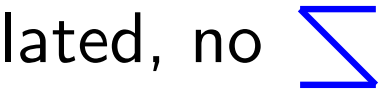
Property: $\tilde{s}_n(v) :=$ analogue of $s_n(v)$ on reduced transversal structures

$$\boxed{s_n(v) = \tilde{s}_n(2+v)}$$ cf 

So $s_n(v)$ has combinatorial meaning for $v \geq -2$

$s_n(-2) = \tilde{s}_n(0) =$ only triangles, no

$\#$ weak rectangulations (Baxter, $\Theta(n^{-4})$)

$s_n(-1) = \tilde{s}_n(1) =$ no  $=$ triangulated, no  cf 

$= \#$ (red) plane bipolar posets $n+2$ vertices

$= \#$ semi-Baxter permutations size n ($\Theta(n^{-6})$)