Rational homotopy theory and computability

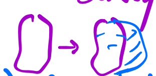
Prelude: "the unreasonable effectiveness of algebraic topology"
(TM Shmuel Weinberger)

Many decision and classification problems in topology can be reduced to homotopy theory:

- vector bundle structures over a space $X \longleftrightarrow [X, BG]$

  set of homotopy classes of maps

  classifying space

- given a manifold M^n, is $M^n = \partial W^{n+1}$? 

  $\Longleftrightarrow$ is a certain map $S^N \to Th(\xi \downarrow Gr)$ homotopic to a constant map?

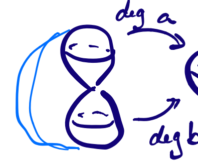
- given a simplicial complex K^k, does K embed in $\mathbb{R}^n$?

  $\Longleftrightarrow$ e.g. when $k \leq \frac{2}{3}n$, an equivariant extension

Fundamental question: are homotopy-theoretic questions decidable?

Examples: (1) (Adian, Rabin) For general Y, $[S^1, Y] = \pi_1(Y)$ is not computable, e.g. "is it 0?" is undecidable (group theory reasons)

(2) (Čadek–Krčál–Matoušek–Vokřínek–Wagner)
For X a 4-dim'l CW cx., subcomplex $A \subset X$,
the extension problem: given a map $f: A \to S^2$, does it extend to X?
is undecidable.

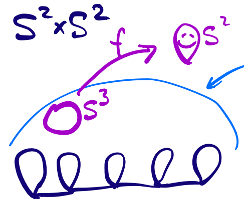


$\alpha \in H^2(S^2)$
$\alpha^2 = 0$
$\Downarrow$
for any $g: X \to S^2$
$(g^*\alpha)^2 = 0$

Why? • $S^2 \vee S^2 \longrightarrow S^2$: $[S^2 \vee S^2, S^2] \cong \mathbb{Z}^2$

- attach a 4-cell to get $S^2 \times S^2$

  $\Longrightarrow 2ab = 0$

- more generally, take  $\vee S^2$

attach a 4-cell
— attaching map gives a quadratic polynomial that's forced to be 0.

- now cut a hole in the 4-cell and fix $f: S^3 \to S^2$

  $\Longrightarrow$ get an equation $\sum$ quadratic terms $= B$. ← Hopf invariant of f

- attach many cells $\Rightarrow$ get a system of eqns

Fact It's undecidable whether such a system has an integer solution.
(special case of Hilbert's 10th problem) (AG/NT reasons)

II. How to build spaces

One way: S^n – space with simplest H^*

adding a cell to X:

attaching map $\in \pi_{n-1}(X)$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r] & X \cup e \end{tikzcd}$$

Another: $K(G,n)$ – space with $\begin{cases} \pi_n = G \\ \pi_i = 0, \ i \neq n \end{cases}$

a Postnikov stage:

contractible space

$$\begin{tikzcd} & \tilde{Y} \arrow[r] \arrow[d] & E(G,n+1) \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & Y \arrow[r] & K(G,n+1) \end{tikzcd}$$

k-invariant $\in H^{n+1}(Y;G)$

Another way of looking at the CKMVW example: S^2 has $\pi_2 \cong \mathbb{Z} \cong \pi_3$

$k_3 = \alpha^2 \in H^4(K(\mathbb{Z},2);\mathbb{Z})$

$\Rightarrow$ so extensions $X \xrightarrow{g} S^2_{(2)} \cong K(\mathbb{Z},2)$ are just classified by $H^2(X,A;\mathbb{Z})$

Note: g lifts to $S^2_{(3)} \iff g^*\alpha^2$ behaving correctly.

morally: when group theory and Hilbert's 10th don't get in the way

III. When is $[X,Y]$ computable?

Theorem: Let Y be a simply connected simplicial complex of finite type, (H_k f.g. $\forall k$)
$d \geq 2$. Then TFAE:

(i) For any simplicial pair (X,A) of cohomological dimension $d+1$ ($H^k(X,A;R) = 0$ for $k > d+1$)
and any simplicial map $f: A \to Y$,
the existence of a continuous extension of f to X is decidable.

(ii) There is a map $Y \xrightarrow{v} \prod_{n=2}^{d} K(G_n, n)$ s.t.
the map $v_*: \pi_n(Y) \to G_n$ has finite kernel and cokernel for $n \leq d$.

Moreover: there is an algorithm which, given simply connected Y
and (X,A) of coh. dim. d, and a simplicial map $f: A \to Y$,

(1) determines whether (ii) is satisfied

(2) if it is, outputs the set $[X,Y]^f$ of homotopy classes rel A
as a (perhaps empty) set on which an abelian group
acts virtually freely and faithfully. ← finite # of orbits each with finite stabilizer

Examples (0) When $\pi_n(Y)$ are all finite (result of Brown '57)

(1) When $d \le 2 \cdot$ (connectivity of Y) (result of CKMVW)

(2) When $Y = S^{2n+1}$ (result of Vokřinek)

(3) When $Y = BG$, G a connected Lie group

$\Rightarrow$ we can compute the set of isomorphism classes of real oriented, complex, or symplectic VB over X

IV. How to think about it?

"only if": Hilbert's 10th problem

"if": Write $H = \prod K(G_n, n)$.

Facts about H: (1) It's an H-space, i.e. there's a map $+ : H \times H \to H$ s.t. the following diagrams are homotopy commutative:

(a)
$$\begin{tikzcd} & H \times H \arrow[dr, "+"] & \\ H \arrow[ur, "{(id,e)}"] \arrow[rr, "id"] \arrow[dr, "{(e,id)}"'] & & H \\ & H \times H \arrow[ur, "+"'] & \end{tikzcd}$$
(identity)

(b)
$$\begin{tikzcd} H \times H \times H \arrow[r, "{(+,id)}"] \arrow[d, "{(id,+)}"'] & H \times H \arrow[d, "+"] \\ H \times H \arrow[r, "+"] & H \end{tikzcd}$$
(associativity)

(c)
$$\begin{tikzcd} H \times H \arrow[d, "{(a,b) \mapsto (b,a)}"'] \arrow[dr, "+"] & \\ H \times H \arrow[r, "+"'] & H \end{tikzcd}$$
(commutativity)

(d)
$$\begin{tikzcd} H \arrow[r, "e"] \arrow[dr, "{a \mapsto (a,-a)}"'] & H \\ & H \times H \arrow[u, "+"'] \end{tikzcd}$$
(homotopy inverse)

(2) This implies that $[X, H]$ is an abelian group with operation $\varphi + \psi$ (pointwise)

(3) Let $\chi_r : H \to H$ take $a \mapsto \underbrace{a + \cdots + a}_{r \text{ times}}$.

For some r, there is a map u s.t.
$$\begin{tikzcd} H \arrow[r, "u"] \arrow[rr, bend right, "\chi_r"'] & Y \arrow[r, "v"] & H \end{tikzcd}$$
homotopy commutes.

$(g \cdot h) \cdot x = g \cdot (h \cdot x)$

(4) This induces an H-space action $act : H \times Y \to Y$, i.e.

$$\begin{tikzcd}
H \times H \times Y \arrow[r, "{(+, id)}"] \arrow[d, "{(id, act)}"'] & H \times Y \arrow[d, "act"] \\
H \times Y \arrow[r, "act"'] & Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
H \times H \arrow[r, "{(id, u)}"] \arrow[d, "+"] & H \times Y \arrow[r, "{(\chi_r, v)}"] \arrow[d, "act"] & H \times H \arrow[d, "+"] \\
H \arrow[r, "u"'] & Y \arrow[r, "v"'] & H
\end{tikzcd}$$

... homotopy commute.

(5) Now choose (X, A) and a map $f : A \to Y$.

The group $[X/A, H]$ acts on $[X, Y]^f$ via

$$[\varphi] \cdot [\alpha] = [act(\varphi, \alpha)].$$