Algorithms and Combinatorics for Geometric Graphs (GeomGraphs)
Homework #2, due November 26th, 2025, AoE

This homework is due November 26th AoE (Anywhere on Earth). Please send me your solutions (or your questions) for this homework exclusively by email at arnaud.de-mesmay@univ-eiffel.fr. You can write either in French or in English.

This homework consists of one long exercise and one smaller, more open-ended exercise. They are independent and can be treated in any order. You can skip a question and use its results in the next questions if you explicitly say so.

This homework is optional and can only contribute positively to your grade. So do not censor yourself and do submit your homework even if it is partial and incomplete.

Exercise 1:

Throughout this exercise we work with orientable surfaces of genus $g > 0$. Recall that a ***triangulation*** of a surface is a cellularly embedded graph where every face has degree exactly three. We say that a triangulation is ***k-cute*** is it has the following two additional properties:

- All the vertices have degree at least k, and
- The graph dual to the triangulation is bipartite, i.e., the triangles can be colored black and white so that adjacent triangles have different colors.

Q1. Show that the torus admits no 8-cute triangulation.

Q2. Show that an 8-cute triangulation has $O(g)$ vertices.

Q3. Show that every orientable surface of genus $g \geq 4$ admits an 8-cute triangulation.[1] *Hint: It is easier to do it with $O(1)$ vertices.*

We now fix a 6-cute triangulation of a surface of genus $g \geq 2$. A ***walk*** on such a triangulation is a finite word $w = v_1 e_1 v_2 e_2 \dots e_{n-1} v_n$ where each edge e_i for $i \in [|1, n-1|]$ has endpoints the vertices v_i and v_{i+1}. A walk is ***trivial*** if $n = 1$ and ***closed*** if $v_k = v_1$. Let w be a walk and $e_{i-1} v_i e_i$ be a vertex of that walk surrounded by two incident edges. We say that w makes a ***k-turn*** at v_i for $k \geq 0$ if w leaves exactly k triangles to its left when going from e_{i-1} to e_i through v. So if w makes a 0-turn at v_i, then $e_{i-1} = e_i$, and if w makes a 1-turn, then e_{i-1} and e_i are consecutive in the clockwise ordering around v_i. Similarly, w makes a ***$-k$-turn*** at v_i for $k \geq 0$ if w leaves exactly k triangles to its right when going from e_{i-1} to e_i through v. Note that a 0-turn is the same as a -0-turn.

[1] If you find 8-cute triangulations for surfaces of genus 2 and 3, do not panic: they do exist but I just phrased the exercise that way because the general construction I have in mind only works for $g \geq 4$.

We say that a turn is ***sharp*** if it is a 0-turn, a 1-turn or a -1-turn, and ***colorblind*** if it is a 2-turn or a -2-turn where e_{i-1} is adjacent to a white triangle on its left and e_i is adjacent to a black triangle on its left. Beware that this definition is not as symmetric as it looks: colorblind turns always look at triangles on the left while -2-turns are defined with respect to triangles on the right. A walk is ***well-behaved*** if it admits neither sharp nor colorblind turns.

Q4. Show that if a walk starts with a 2-turn, then continues with an arbitrary number of 3-turns and finishes with a 2-turn, then exactly one of the two 2 turns is colorblind.

Q5. Let w be a closed walk so that S cut along w consists of exactly two components, and the component to the left of w is a topological disk D. So in particular w is simple, i.e., it has no repeated vertices nor edges. Denote by n_k the number of vertices on w which lie on the boundary of the disk and are incident to exactly k triangles of D. Show that $2n_1 + n_2 \geq 6 + \sum_{k \geq 4} n_k$.

Q6. Deduce from the two previous questions that there are at least three vertices in the walk w that make a sharp or a colorblind turn.

A walk can naturally be considered as a curve on a surface, i.e., a map $w : [0, 1] \to S$. Two walks w_1 and w_2 with the same endpoints u and v are ***homotopic*** if there exists a continuous map $h : [0, 1] \times [0, 1] \to S$ so that $h(\cdot, 0) = w_1$, $h(\cdot, 1) = w_2$ and for all t in $[0, 1]$, $h(0, t) = u$ and $h(1, t) = v$. Thus a homotopy is a continuous deformation between w_1 and w_2 which preserves their endpoints. The point of the exercise is the following two questions which are a bit harder than the previous ones.

Q7*. Show that in a 6-cute triangulation, any two homotopic well-behaved walks are equal.

Q8*. Show that in a 6-cute triangulation, any walk is homotopic to a unique well-behaved walk. Provide a polynomial-time algorithm to compute it. *Hint: You can first try some homotopies to locally circumvent sharp and colorblind turns and see where it leads you.*

Exercise 2:

We say that a simple closed curve γ on a non-orientable surface S of genus $g \geq 1$ is ***correcting*** if cutting S along γ yields exactly one connected component, and that component is an orientable surface with boundary. Let G be a graph that is cellularly embedded on a non-orientable surface of genus $g \geq 1$. Provide a polynomial-time that computes a correcting curve in general position with respect to G. There are bonus points if you prove an upper bound[2] on the number of intersections of the curve with G in terms of the number of edges of G, and even more bonus points if you prove that your upper bound is optimal.

[2]The lower the better but do not bother with the constants and focus on the asymptotics $O(f(|E(G)|)$.