Exercises on embedded graphs, Lecture 8

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Exercise 1:

Recall that a simple closed curve on a surface S is an embedding of the circle $\mathbb{S}^1 \hookrightarrow S$. Therefore, any two simple closed curves are homeomorphic since they are all homeomorphic to $\mathbb{S}^1$. However, it makes sense to ask whether two simple closed curves are congruent by a homeomorphism of the surface, which we abusedly also call *homeomorphic* (there is no risk of confusion since the other definition is vacuous).

1. Show that on an orientable surface of genus $g > 0$, there are exactly $\lfloor g/2 \rfloor + 2$ homeomorphism classes of simple closed curves.
2. What about non-orientable surfaces?

Solution: It is important for this exercise to *not* try to imagine what a homeomorphism can look like. Even up to homotopy, this is a complex topic, see for example here. Instead, one can just rely on the classification of surfaces with boundary.

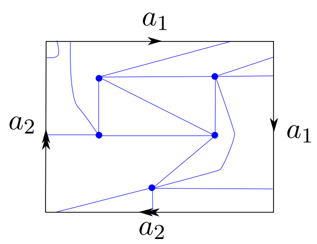
1. In the orientable case, when one cuts a surface of genus g along a simple non-contractible non-separating curve, one obtains a surface with one connected component (the curve is non-separating), two boundaries (each side of the curve), orientable (cutting does not create orientation-reversing gluings) and of genus $g - 1$ (as you can check by computing the Euler characteristic). There is exactly one such surface up to homeomorphism by the classification of surfaces with boundary. This homeomorphism carries over when one glues back the surface. Therefore, any two simple non-contractible non-separating curves are homeomorphic. For separating curves, we obtain two orientable surfaces of genus g_1 and g_2 with one boundary each, where $g_1, g_2 \geq 0$ and $g_1 + g_2 = g$ (as you can check again by computing the Euler characteristic): these are classified depending on how the genus g splits in g_1 and g_2: there are exactly $\lfloor g/2 \rfloor + 1$ choices.
2. In the non-orientable case, the fauna is richer: there are more cases depending on whether cutting along the curve yields one or two boundaries and whether the resulting surface(s) is(are) orientable. The second exercise of the homework encourages you to think over the most mysterious case: correcting curves such that cutting along them yields a single orientable component. Are they all homeomorphic? The remaining cases are:
   - One-sided non-separating non-correcting curve: after cutting one gets a single orientable surface with one boundary component. This is for example a in the surface $aabbccdd$. (if this troubles you, make a Mobius band out of paper, cut along the middle curve and notice that this only adds one boundary and not two!)
   - Two-sided non separating non-correcting curve: after cutting one gets a single orientable surface with two boundary component. This is for example a in the surface $ab\bar{a}\bar{b}ccdd$.

- Curves cutting in two non-orientable components of genus g_1 and g_2 such that $g_1 + g_2 = g$. For this you can take the relevant diagonal in the polygonal scheme *aabbccdd*.
- Curves cutting in one non-orientable component and one orientable component of genus g_1 and g_2 such that $g_1 + 2g_2 = g$. The case $g_2 = 0$ is the case of contractible curves. For this you can take the relevant diagonal in the polygonal scheme $a_1b_1\bar{a_1}\bar{b_1}\dots a_{g_1}b_{g_1}\bar{a_{g_1}}\bar{b_{g_1}}c_1c_1\dots c_{g_2}c_{g_2}$.

Note that separating curves always yield two boundaries (one in each component) and that gluing two orientable surfaces always yields an orientable surface, so these are all the possible cases.

Exercise 2 (already in the previous sheet but I like it):

Recall that a cellular embedding is an embedding where all the faces are disks, and that a non-orientable surface of genus g is a surface with polygonal scheme $a_1a_1a_2a_2\dots a_ga_g$. A convenient way to represent a graph on a non-orientable surface is to draw it on top of this polygonal scheme. For example, here is a cellular embedding of K_5 on a non-orientable surface of genus two.



1. Provide an explicit cellular embedding of K_4 on a non-orientable surface of genus 3.
2. Let G be a simple graph with v vertices, e edges cellularly embedded on a non-orientable surface of genus g. Prove that $g \leq e - v + 1$.
3. Let G be a simple graph with v vertices and e edges, and let g_1 be the smallest genus of a non-orientable surface on which G embeds. Prove that for any g such that $g_1 \leq g \leq e - v + 1$, G can be cellularly embedded on a non-orientable surface of genus g.
4. In particular, G can always be cellularly embedded on a non-orientable surface of genus $e - v + 1$. Provide a linear-time algorithm to compute such an embedding.

Exercise 3:

A **cut-graph** is a graph C embedded on a surface S so that cutting S along C yields a disk.

1. Provide a linear-time algorithm that, given as input a graph G cellularly embedded on an orientable surface of genus g, outputs a subgraph C of G that is a cut-graph.

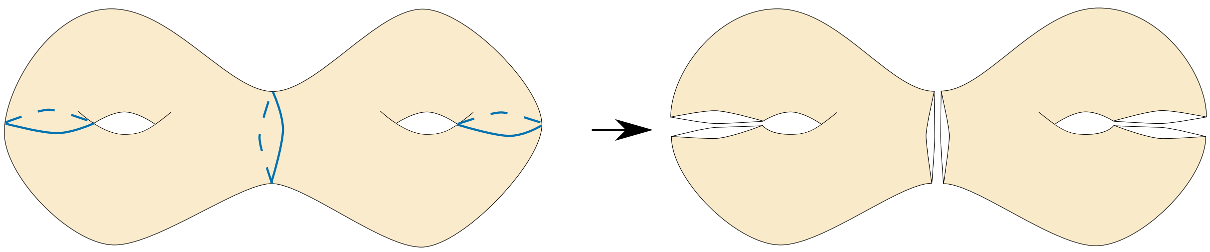
Given a cut-graph C, we define a **reduced cut-graph** by inductively applying the following two operations until none of them are available:

- If there is a vertex of degree 1, remove it and its adjacent edge.
- If there is a vertex v of degree 2, **dissolve** it, that is, replace its two adjacent edges uv and vw with a single edge uw.

2. Show that a reduced cut-graph on an orientable surface of genus g has $O(g)$ vertices and $O(g)$ edges.

A **pair of pants** is a surface homemorphic to a sphere with 3 boundaries. The picture below explains this somewhat surprising name.

A **pants decomposition** of an orientable surface S of genus g is a family of closed curves $\Gamma = (\gamma_1, \ldots, \gamma_k)$ such that cutting S along Γ yields a family of surfaces with boundary, each of which is homeomorphic a pair of pants. The figure below pictures a pants decomposition of an orientable surfaces of genus 2.



3. Show that a pants decomposition of a surface of genus g contains exactly $3g - 3$ curves. *Hint: You can add a well-chosen family of vertices and edges to a pants decomposition in order to obtain a cellularly embedded graph*

4. Provide a polynomial-time algorithm that, given as input a graph G cellularly embedded on an orientable surface S of genus g, computes a family of curves in general position with respect to G that forms a pants decomposition of S.