

Exercise 1. Some basic results.

Let G be a graph cellularly embedded in the plane. Let v , e , and f be the number of vertices, edges, and faces of G .

1. Assume that G is simple (has no loops and no multiple edges), so that in particular each face of G has degree at least three. Prove that $e = O(v)$.
2. Prove that G has either a vertex or a face of degree at most three.
3. Prove that G has either a vertex of degree at most six or a face of degree at most two. In particular, if G is simple, then it has a vertex of degree at most six.

Exercise 2. A polygonal version of the Jordan curve theorem.

A graph G is *straight-line embedded* in the plane if it is embedded and each edge is drawn as a straight-line segment.

1. Let F be a forest straight-line embedded in $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus F$ is connected.

Now, let P be a polygon (without self-crossings) in $\mathbb{R}^2$. Our goal is to prove that $\mathbb{R}^2 \setminus P$ has exactly two connected components. We assume without loss of generality that P has no horizontal edge.

2. Let us prove that $\mathbb{R}^2 \setminus P$ has at least two connected components. For each point p in $\mathbb{R}^2 \setminus P$ whose y -coordinate is different from a vertex of P , define $\gamma(p)$ to be the number of intersection points of the indefinite horizontal ray starting at p and going rightwards, modulo 2.
 - (a) Prove that γ can be extended to a continuous map from $\mathbb{R}^2 \setminus P$ to $\{0, 1\}$.
 - (b) Prove that γ is onto.
 - (c) Conclude.
3. Let us prove that $\mathbb{R}^2 \setminus P$ has at most two connected components. Let $\varepsilon > 0$ be simultaneously smaller than:
 - the distance between any two vertices of P ,
 - the distance between any vertex of P and any non-incident edge of P ,
 - the distance between any two edges of P that do not share a vertex, and
 - the length of any edge of P .

Let

$$U = \{x \in \mathbb{R}^2 \mid 0 < d(x, P) < \varepsilon/100\}.$$

- (a) Prove that each connected component of $\mathbb{R}^2 \setminus P$ contains at least one point of U .
- (b) Prove that U has at most two connected components.
- (c) Conclude.

Exercise 3. Weights of enclosing circuits.

Let G be a graph cellularly embedded in the plane, specified by its combinatorial map. We assume that each face has a real *weight*.

Let T be a spanning tree of G . Given an edge e in $G - T$, let $\gamma(e)$ be the circuit of G that contains e together with the unique path in T between its endpoints.

Give an algorithm, as efficient as possible, that, for *all* edges e of $G - T$, computes the sum of the weights of all faces inside $\gamma(e)$.