Master 2 Mathematics and Computer Science
Symbolic Dynamics. Lecture 6

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Recognizability

Let $\sigma\colon A^* \to B^*$ be a substitution. A *σ-representation* of $y \in B^{\mathbb{Z}}$ is a pair (x, k) of a sequence $x \in A^{\mathbb{Z}}$ and an integer k such that

$$y = S^k(\sigma(x)). \tag{1}$$

The σ-representation (x, k) is *centered* if $0 \le k < |\sigma(x_0)|$.

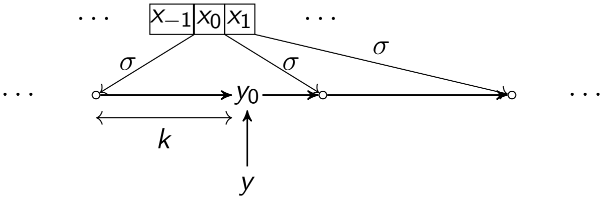


Figure: A centered σ-representation (x, k) of y.

Note, in particular, that a centered σ-representation (x, k) is such that $\sigma(x_0) \neq \varepsilon$.

Note that if y has a (not necessarily centered) σ-representation (x, ℓ), then it has also a centered σ-representation (x', k), where x' a shift of x.

Indeed, assume $\ell \geq 0$ (the case $\ell < 0$ is symmetric). Let $i \geq 0$ be such that $|\sigma(x_0 \cdots x_{i-1})| \leq \ell < |\sigma(x_0 \cdots x_i)|$. Set $k = \ell - |\sigma(x_0 \cdots x_{i-1})|$ and $x' = S^i x$. Then $S^k\sigma(x') = S^{k+|\sigma(x_0 \cdots x_{i-1})|}\sigma(x) = S^\ell\sigma(x) = y$ and $0 \leq k < |\sigma(x'_0)|$. Thus, (x', k) is a centered σ-representation of y.

For a shift space X on A, the set of points in $B^{\mathbb{Z}}$ having a σ-representation (x, k) with $x \in X$ is a shift space on B, which is the closure under the shift of $\sigma(X)$.
Indeed, if (x, k) is a σ-representation of y, then $S(y)$ has the σ-representation (x', k') with

$$(x', k') = \begin{cases} (x, k+1) & \text{if } k+1 < |\sigma(x_0)| \\ (S(x), 0) & \text{otherwise.} \end{cases}$$

Let X be a shift space on A.
The substitution $\sigma\colon A^* \to B^*$ is *recognizable* in X if every $y \in B^{\mathbb{Z}}$ has **at most one** centered σ-representation (x, k) such that $x \in X$.

Thus, in informal terms, for a sequence y on B, there is at most one way to desubstitute y to obtain a sequence in X.

Example

The substitution $\sigma\colon a \mapsto a, b \mapsto ab, c \mapsto abb$ is recognizable in the full shift $X = \{a, b, c\}^{\mathbb{Z}}$.
Indeed, let Y be the closure under the shift of $\sigma(X)$.
Any two consecutive occurrences of a are separated by a block of zero, one or two b, which determines the rule of σ to be used for desubstitution. Formally, we have

$$\begin{aligned}
\sigma([a]_X) &= [aa]_Y, \\
\sigma([b]_X) &= [aba]_Y, \quad S\sigma([b]_X) = [a \cdot ba]_Y \\
\sigma([c]_X) &= [abba]_Y, \quad S\sigma([c]_X) = [a \cdot bba]_Y, \quad S^2\sigma([c]_X) = [ab \cdot ba]_Y
\end{aligned}$$

and these sets form a partition of Y.

Example

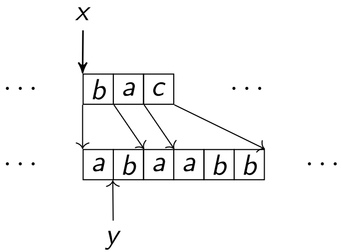


Figure: A centered σ-representation of $y = \cdots a \cdot baabb \cdots$.

Fully recognizable substitutions

Let $\sigma\colon A^* \to B^*$ be a substitution. Assume that σ is erasing, but that not all letters are erasable. Then σ cannot be recognizable in $A^{\mathbb{Z}}$. Indeed, if $\sigma(a) \neq \varepsilon$, and $\sigma(b) = \varepsilon$, then $\sigma({}^{\omega}ab \cdot a^{\omega}) = \sigma(a^{\infty})$.

Let $\sigma\colon A^* \to B^*$ be a non-erasing substitution. We say that σ is *fully recognizable* or *circular* if it is recognizable in $A^{\mathbb{Z}}$.

Thus, in particular, a circular substitution is injective.

Example

Example

The substitution $\sigma\colon a \mapsto a, b \mapsto ab, c \mapsto abb$ is fully recognizable.

A *coding substitution* for a set U of nonempty words on A is a substitution $\phi \colon B^* \to A^*$ such that its restriction to B is a bijection onto U. The set U is called a *code* if ϕ is injective and a *circular code* if ϕ is circular.

Proposition

Let X be a minimal shift space on A and let $u \in \mathcal{B}(X)$. Any coding substitution $\phi \colon B^* \to A^*$ for the set $\mathcal{R}_X(u)$ of return words to u is circular.

Proof.

Since wu contains exactly two occurrences of u for each $w \in \mathcal{R}_X(u)$, for each $y \in X$, there is a unique sequence $z = \cdots w_{-1} \cdot w_0 w_1 \cdots$ with $w_i \in \mathcal{R}_X(u)$, and a unique integer k such that $y = S^k(z)$ with $0 \le k < |w_0|$. Since ϕ is a coding substitution, for each $w_i \in \mathcal{R}_X(u)$, there is a unique $b_i \in B$ such that $\phi(b_i) = w_i$. Hence, there is a unique $x \in B^{\mathbb{Z}}$ and k with $0 \le k < |\phi(x_0)|$ such that $y = S^k\phi(x)$. $\square$

Representability

Existence of a representation

Proposition

Let $\sigma\colon A^* \to A^*$ be a substitution. Every point y in $\mathsf{X}(\sigma)$ has a σ-representation $y = S^i(\sigma(x))$ for some $i \geq 0$, and x in $\mathsf{X}(\sigma)$.

Existence of a representation

Proof.

Let $k = |\sigma|$ and let y be in $\mathrm{X}(\sigma)$. For every $n \geq 1$, there is an integer $m \geq 1$ such that $y_{[-n,n]}$ occurs in $\sigma^m(a)$ for some letter $a \in A$.
For every $n > 2k$, there is an integer $0 \leq i \leq k$ such that, for an infinity of $n > 2k$, there are words u_n, v_n with $u_n v_n \in \mathcal{L}(\sigma)$ such that $y_{[-n+k,-i)}$ is a suffix of $\sigma(u_n)$ and $y_{[-i,n-k]}$ is a prefix of $\sigma(v_n)$. Further, $|u_n| \geq (n-k-i)/k$ and $|v_n| \geq (n-k+i)/k$.
Therefore, there are infinitely many $n > 2k$ for which the value of i is the same.
By a compactness argument, we get that there is a point $x \in X(\sigma)$ such that $y = S^i(\sigma(x))$. □

Elementary substitutions

Elementary substitution

A substitution $\sigma\colon A^* \to C^*$ is *elementary* if for every alphabet B and every pair of substitutions $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ such that $\sigma = \alpha \circ \beta$, one has $\mathrm{Card}(B) \geq \mathrm{Card}(A)$.

In this case, one has in particular $\mathrm{Card}(C) \geq \mathrm{Card}(A)$.

Moreover, σ is non-erasing (Exercise).

Example

Example

The Thue-Morse substitution $\sigma\colon a \mapsto ab, b \mapsto ba$ is elementary. Indeed, if $\sigma = \alpha \circ \beta$ with $\beta\colon \{a, b\}^* \to c^*$, then $ab = \alpha(c^i)$ and $ba = \alpha(c^j)$ which is impossible.

Example

The substitution $\sigma\colon a \mapsto ab, b \mapsto abc, c \mapsto cc$ is not elementary. Indeed, we have $\sigma = \alpha \circ \beta$ with $\alpha\colon u \mapsto ab, v \mapsto c$ and $\beta\colon a \mapsto u, b \mapsto uv, c \mapsto vv$.

Note that the property of being elementary is decidable.
Indeed, if $\sigma \colon A^* \to C^*$ is a substitution consider the finite family $\mathcal{F}$ of sets $U \subset C^*$ such that $\sigma(A) \subset U^* \subset C^*$ with every $u \in U$ occurring in some $\sigma(a)$ for $a \in A$.
Then σ is elementary if and only if $\mathrm{Card}(U) \geq \mathrm{Card}(A)$ for every $U \in \mathcal{F}$.

Elementary substitution

Proposition

Let $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} C^*$ *be substitutions. If* $\alpha \circ \beta$ *is elementary, then* β *is elementary.*

Proof.

Let $A^* \xrightarrow{\gamma} D^* \xrightarrow{\delta} B^*$ be such that $\beta = \delta \circ \gamma$. Then $\alpha \circ \beta = \alpha \circ (\delta \circ \gamma) = (\alpha \circ \delta) \circ \gamma$. This implies $\mathrm{Card}(D) \geq \mathrm{Card}(A)$. Thus β is elementary. □

Elementary substitution

A sufficient condition for a substitution to be elementary can be formulated in terms of its composition matrix.

Proposition

If the rank of $M(\sigma)$ is equal to $\mathrm{Card}(A)$, then σ is elementary.

Proof.

Indeed, if $\sigma = \alpha \circ \beta$ with $\beta \colon A^* \to B^*$ and $\alpha \colon B^* \to C^*$, then $M(\sigma) = M(\alpha)M(\beta)$. If $\mathrm{rank}(M(\sigma)) = \mathrm{Card}(A)$, then

$$\mathrm{Card}(A) = \mathrm{rank}(M(\sigma)) \leq \mathrm{rank}(M(\alpha)) \leq \mathrm{Card}(B).$$

Thus σ is elementary. $\square$

This condition is not necessary. For example, the Thue-Morse substitution $\sigma \colon a \mapsto ab, b \mapsto ba$ is elementary, but its composition matrix has rank one.

Elementary substitution

If $\sigma\colon A^* \to C^*$ is a substitution, we define

$$\ell(\sigma) = \sum_{a \in A} (|\sigma(a)| - 1). \tag{2}$$

We say that a decomposition $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ is *trim* if

(i) α is non-erasing,

(ii) for each $b \in B$ there is an $a \in A$ such that $\beta(a)$ contains b.

Proposition

Let $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ be a trim decomposition of σ. Then

$$\ell(\alpha \circ \beta) \geq \ell(\alpha) + \ell(\beta). \tag{3}$$

Proof.

Set $\sigma = \alpha \circ \beta$. We have

$$\begin{aligned}
\ell(\sigma) - \ell(\beta) &= \sum_{a \in A} (|\sigma(a)| - |\beta(a)|) \\
&= \sum_{a \in A} \sum_{b \in B} (|\alpha(b)||\beta(a)|_b - |\beta(a)|_b) \\
&= \sum_{a \in A} \sum_{b \in B} (|\alpha(b)| - 1)||\beta(a)|_b \\
&= \sum_{b \in B} ((|\alpha(b)| - 1) \sum_{a \in A} |\beta(a)|_b).
\end{aligned}$$

Since every b occurs in some $\beta(a)$, every factor $\sum_{a \in A} |\beta(a)|_b$ is positive, whence the conclusion. □

Elementary substitution

Proposition

An elementary substitution $\sigma\colon A^* \to C^*$ is injective on $A^{\mathbb{N}}$.

follows from:

Proposition

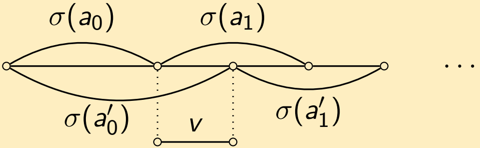
If a substitution $\sigma\colon A^* \to C^*$ is not injective on $A^{\mathbb{N}}$, there is a trim decomposition $\sigma = \alpha \circ \beta$ with $\alpha\colon B^* \to C^*$ and $\beta\colon A^* \to B^*$ such that α is injective on $B^{\mathbb{N}}$, $\mathrm{Card}(B) < \mathrm{Card}(A)$ and every $b \in B$ occurs as the first letter of $\beta(a)$ for some $a \in A$.

Proof

Proof.

Assume first that σ is non-erasing. We use an induction on $\ell(\sigma)$. If $\ell(\sigma) = 0$, set $B = \sigma(A)$. Let α be the identity on B^* and let $\beta = \sigma$. All conditions are clearly satisfied.

Assume now that the statement is true for $\ell < \ell(\sigma)$. Since σ is not injective on $A^{\mathbb{N}}$, we have $\sigma(a_0 a_1 \cdots) = \sigma(a'_0 a'_1 \cdots)$ for some $a_i, a'_i \in A$ with $a_0 \neq a'_0$. We can assume that $\sigma(a_0)$ is a prefix of $\sigma(a'_0)$. Set $\sigma(a'_0) = \sigma(a_0)v$. If v is empty, set $B = A \setminus \{a_0\}$. Let α be the restriction of σ to B and let β be defined by $\beta(a'_0) = a_0$ and $\beta(a) = a$ for $a \neq a'_0$. Clearly, $\sigma = \alpha \circ \beta$, and all conditions are satisfied.



Proof

Proof.

Next, assume that v is nonempty. Define $\alpha_1 \colon A^* \to C^*$ by $\alpha_1(a'_0) = v$ and $\alpha_1(a) = \sigma(a)$ for $a \neq a'_0$. Next, define $\beta_1 : A^* \to A^*$ by $\beta_1(a'_0) = a_0 a'_0$ and $\beta_1(a) = a$ for $a \neq a'_0$. Then $\sigma = \alpha_1 \circ \beta_1$ since

$$\alpha_1 \circ \beta_1(a'_0) = \alpha_1(a_0 a'_0) = \sigma(a_0)v = \sigma(a'_0),$$

and $\alpha_1 \circ \beta_1(a) = \alpha_1(a) = \sigma(a)$ if $a \neq a'_0$. The substitution β_1 is injective on $A^{\mathbb{N}}$ because no word in $\beta_1(A)$ begins with a'_0. Thus α_1 is not injective on $A^{\mathbb{N}}$. By Equation (3), since the decomposition is trim, we have $\ell(\alpha_1) < \ell(\sigma)$. By induction hypothesis, we have a decomposition $\alpha_1 = \alpha_2 \circ \beta_2$ for $\beta_2 : A^* \to B^*$ and $\alpha_2 : B^* \to C^*$ with $\mathrm{Card}(B) < \mathrm{Card}(A)$, the substitution α_2 being injective on $B^{\mathbb{N}}$ and every letter $b \in B$ occurring as initial letter in the word $\beta_2(a)$ for some $a \in A$. Note that, since α_1 is non-erasing, β_2 is non-erasing. □

Proof.

Set $\beta = \beta_2 \circ \beta_1$. Since β_1, β_2 are non-erasing, β is non-erasing. Then $\sigma = \alpha_1 \circ \beta_1 = \alpha_2 \circ \beta_2 \circ \beta_1 = \alpha_2 \circ \beta$. The decomposition $\sigma = \alpha_2 \circ \beta$ satisfies all the required conditions.

Indeed, let $b \in B$. Then there is $a \in A$ such that b is the first letter of $\beta_2(a)$.

If $a \neq a'_0$, we have $\beta_1(a) = a$ and thus b is the first letter of $\beta(a)$.

Suppose next that $a = a'_0$. Since $\sigma(a_0 a_1 \cdots) = \sigma(a'_0 a'_1 \cdots)$ and since α_2 is injective on $B^{\mathbb{N}}$, we have $\beta(a_0 a_1 \cdots) = \beta(a'_0 a'_1 \cdots)$.

Since $\beta_1(a_0) = a_0$ and $\beta_1(a'_0) = a_0 a'_0$, we obtain $\beta_2(a_0)\beta(a_1 \cdots) = \beta_2(a_0 a'_0)\beta(a'_1 \cdots)$ and thus

$$\beta(a_1 \cdots) = \beta_2(a'_0)\beta(a'_1 \cdots),$$

showing, since β is non-erasing, that b is the initial letter of $\beta(a_1)$. $\square$

Proof.

Now consider a substitution σ such that the set $B = \{a \in A \mid \sigma(a) \neq \varepsilon\}$ is strictly contained in A. Let $\beta\colon A^* \to B^*$ be defined by $\beta(a) = a$ if $a \in B$ and $\beta(a) = \varepsilon$ otherwise. Let α be the restriction of σ to B^*. Then $\sigma = \alpha \circ \beta$ and α is non-erasing. If α is injective on $B^{\mathbb{N}}$, we are done. Otherwise, by the first part of the proof, we have $\alpha = \alpha_1 \circ \beta_1$ with $\alpha_1\colon B_1^* \to C^*$ and $\beta_1\colon B^* \to B_1^*$ with α_1 injective on $B_1^{\mathbb{N}}$, $\mathrm{Card}(B_1) < \mathrm{Card}(B)$ and every $b_1 \in B_1$ occurs as the first letter of some $\beta_1(b)$. Then the decomposition $\sigma = \alpha_1 \circ (\beta_1 \circ \beta)$ satisfies all the conditions. $\square$

By a symmetric version, an elementary substitution $\sigma\colon A^* \to C^*$ is injective on $A^{-\mathbb{N}}$. Since a substitution which is injective on $A^{\mathbb{N}}$ and on $A^{-\mathbb{N}}$ is injective on $A^{\mathbb{Z}}$, we obtain the following corollary of Proposition 6.

Proposition

An elementary substitution $\sigma\colon A^* \to C^*$ is injective on $A^{\mathbb{Z}}$.

Recognizability for aperiodic points

A substitution $\sigma\colon A^* \to B^*$ is *recognizable in X for aperiodic points* if **every aperiodic point** $y \in B^{\mathbb{Z}}$ has at most one centered representation **in** X.

We say that σ is *fully recognizable for aperiodic points* if it is recognizable in the full shift for aperiodic points.

Example

Example

The substitution $\sigma\colon a \mapsto aa, b \mapsto ab, c \mapsto ba$ is not fully recognizable for aperiodic points.
Indeed, every sequence without occurrence of bb has two factorizations in words of $\{aa, ab, ba\}$.

Proposition

The family of substitutions that are fully recognizable for aperiodic points is closed under composition.

Aperiodic substitution

A substitution σ is *aperiodic* if $X(\sigma)$ contains no periodic point.

Theorem (B. Mossé 1992, B. Mossé 1996)

Any aperiodic substitution is recognizable in $X(\sigma)$.

Recognizability for aperiodic points

Theorem (J. Karhumäki, J. Maňuch, W. Plandowski 2003)

An elementary substitution is fully recognizable for aperiodic points.

Proof

A substitution $\sigma\colon A^* \to B^*$ with no erasable letter is *left-marked* if each word $\sigma(a)$, for $a \in A$, begins with a distinct letter.

In particular, if σ is left-marked, σ is injective on A and $\sigma(A)$ is a prefix code.

It is clear that a left-marked substitution is elementary.

Left-marked substitution

Proposition

If $\sigma : A^* \to B^*$ is left-marked, then it is fully recognizable for aperiodic points.

Proof.

Assume that $y \in B^{\mathbb{Z}}$ has two distinct σ-representations (x, k) and (x', k'). We may assume $k = 0$. We will prove that y is periodic. Let P be the set of proper prefixes of the elements of $U = \sigma(A)$. For $p \in P$ and $a \in A$, there is at most one $q \in P$ such that $p\sigma(a) \in U^*q$. We write $q = p \cdot a$ when such a q exists. Let $p_0 = y_{-k'} \cdots y_{-1}$ (with $p_0 = \varepsilon$ if $k' = 0$). Since $y = \sigma(x) = S^{k'}(\sigma(x'))$, we have (see Figure)

$$\sigma(\cdots x'_{-2}x'_{-1})p_0 = \sigma(\cdots x_{-1}), \quad p_0\sigma(x_0x_1\cdots) = \sigma(x'_0x'_1\cdots).$$

As a consequence, there exists, for each $n \in \mathbb{Z}$, a word $p_n \in P$ such that $p_n \cdot x_n = p_{n+1}$.

□

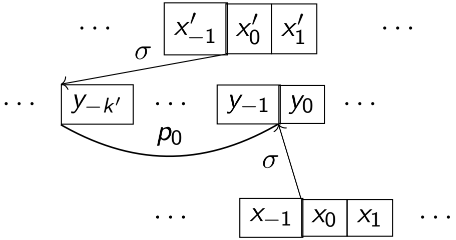


Figure: The two centered σ-representations of y.

Proof.

Consider the labeled graph G with P as set of vertices and edges (p, a, q) if $p \cdot a = q$. Since σ is left-marked, there is for every nonempty $p \in P$ at most one $a \in A$ such that $p \cdot a$ exists. In particular, since all edges going out of ε end in ε, G is a disjoint union of simple cycles in $P \setminus \{\varepsilon\}$ and loops on ε. As a consequence, either the path is a cycle, and thus x and y are periodic, or $k' = 0$ and thus $x = x'$ since σ is left-marked. $\square$

Example

The Thue-Morse substitution $\sigma : a \to ab, b \to ba$ is left-marked. Thus, it is fully recognizable for aperiodic points. The graph used in the proof is:

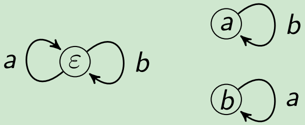


Figure: The graph associated with the Thue-Morse substitution.

An elementary substitution is fully recognizable for aperiodic points

Proof.

Let $\sigma\colon A^* \to B^*$ be an elementary substitution. We use an induction on $\ell(\sigma) = \sum_{a\in A}(|\sigma(a)| - 1)$ (see (2)). Since σ is elementary, it has no erasable letter, and the minimal possible value of $\ell(\sigma)$ is 0. In this case, σ is a bijection from A into B, and thus it is fully recognizable.

Assume now that σ is not fully recognizable for aperiodic points. Thus, there exist $x, x' \in A^{\mathbb{Z}}$, $a' = x'_0 \in A$ and w with $0 < |w| < |\sigma(x_0)|$ such that $\sigma(x) = w\sigma(x')$ for some proper suffix w of $\sigma(a')$. Set $\sigma(a') = vw$ (see Figure). We can then write $\sigma = \sigma_1 \circ \tau_1$ with $\tau_1\colon A^* \to A_1^*$ and $\sigma_1\colon A_1^* \to B^*$ and $A_1 = A \cup \{a''\}$ where a'' is a new letter. We have $\tau_1(a') = a'a''$ and $\tau_1(a) = a$ otherwise. In particular, τ_1 is left-marked. Next $\sigma_1(a') = v$, $\sigma_1(a'') = w$ and $\sigma_1(a) = \sigma(a)$ otherwise. Since $\ell(\tau_1) > 0$, we have $\ell(\sigma_1) < \ell(\sigma)$ by Equation (3). □

Proof.

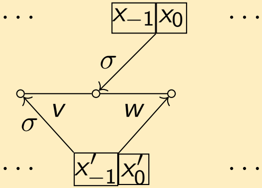


Figure: The words v, w.

Since

$$\sigma_1(a'')\sigma_1(\tau_1(x')) = \sigma_1(\tau_1(x)),$$

and since $\tau_1(x_0)$ does not begin with a'', σ_1 is not injective on $A_1^{\mathbb{N}}$. $\square$

Proof.

By Proposition 7, we can write $\sigma_1 = \sigma_2 \circ \tau_2$ with $\sigma_2 : A_2^* \to B^*$ and $\tau_2 : A_1^* \to A_2^*$ for some alphabet A_2 such that $\mathrm{Card}(A_2) < \mathrm{Card}(A_1)$ and that every letter $c \in A_2$ appears as the first letter of some $\tau_2(a)$ for $a \in A_1$. Then, by Equation (3), we have

$$\ell(\sigma_1) \geq \ell(\sigma_2) + \ell(\tau_2). \tag{4}$$

Moreover, since σ_1 is non-erasing, τ_2 is non-erasing and thus $\ell(\tau_2) \geq 0$. This implies $\ell(\sigma_1) \geq \ell(\sigma_2)$.
Since σ is elementary, we have $\mathrm{Card}(A_2) \geq \mathrm{Card}(A)$. Since $\mathrm{Card}(A_2) < \mathrm{Card}(A_1) = \mathrm{Card}(A) + 1$, this forces $\mathrm{Card}(A_2) = \mathrm{Card}(A)$. We may also assume that σ_2 and $\tau_2 \circ \tau_1$ are elementary since otherwise σ is not elementary.

□

Proof.

Since σ_2 is elementary and since $\ell(\sigma_2) \leq \ell(\sigma_1) < \ell(\sigma)$, by the induction hypothesis, σ_2 is fully recognizable for aperiodic points. The decomposition $\sigma = \sigma_1 \circ (\tau_2 \circ \tau_1)$ is trim. Indeed, σ_2 is elementary and thus non-erasing. Next, every letter of A_2 appears in some $\tau_2(a)$ and, by definition of τ_1, it appears also in some $\tau_2 \circ \tau_1(a)$. Thus, we have also

$$\ell(\sigma) \geq \ell(\sigma_2) + \ell(\tau_2 \circ \tau_1). \tag{5}$$

Thus, if $\ell(\sigma_2) > 0$, the inequality $\ell(\tau_2 \circ \tau_1) < \ell(\sigma)$ holds. Since $\tau_2 \circ \tau_1$ is elementary, we obtain that $\tau_2 \circ \tau_1$ is fully recognizable for aperiodic points by induction hypothesis. Since the family of substitutions that are fully recognizable for aperiodic points is closed under composition, we get that that σ is fully recognizable for aperiodic points. □

Proof.

Let us finally assume that $\ell(\sigma_2) = 0$. Since $\sigma_1(a'') = w$ is a prefix of $\sigma(x_0) = \sigma_1(\tau_1(x_0)) = \sigma_1(x_0)$ with $x_0 \in A$, and since σ_2 is a bijection from A_2 onto B, the first letter of $\tau_2(a'')$ is equal to the first letter of $\tau_2(x_0)$. Further, each letter of A_2 appears as the first letter of $\tau_2(c)$ for some letter $c \in A_1$. Thus, each letter of A_2 is the first letter of $\tau_2(a)$ for some letter $a \in A$. Consequently, each letter of B is the first letter of $\sigma(a)$ for some $a \in A$. Since $\mathrm{Card}(A) = \mathrm{Card}(B)$, it follows that σ is left-marked.

We obtain the conclusion by the proposition for left-marked substitutions. $\square$

Recognizability for aperiodic points

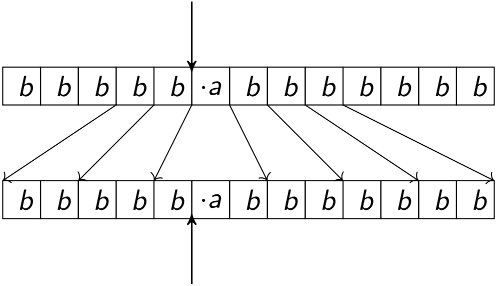
Recognizability for aperiodic points

Theorem (Berthé et al. 2018 for non-erasing substitutions, B. et al. 2022)

Any morphism $\sigma \colon A^* \to A^*$ *is recognizable for aperiodic points in* $X(\sigma)$.

Example

Let $\sigma\colon a \mapsto bab, b \mapsto bb$.

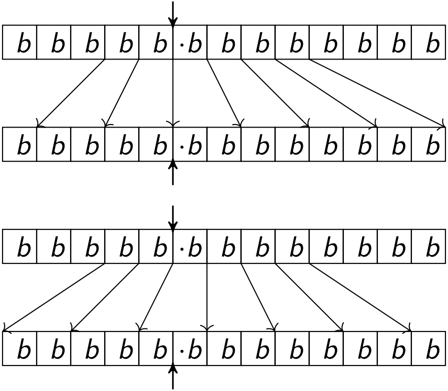


The point $y = \cdots bbbb \cdot abbbb \cdots = S(\sigma(y))$ has a unique centered σ-representation $(y, 1)$.

Example

Example

Let $\sigma\colon a \mapsto bab,\ b \mapsto bb$.

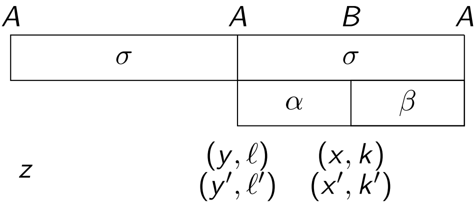


The point $y = \cdots bbbb \cdot bbbbb \cdots = \sigma(y) = S(\sigma(y))$ has a two centered σ-representation $(y, 0)$ and $(y, 1)$.

Lemma

Let $\sigma \colon A^* \xrightarrow{\sigma} A^*$ *be a substitution and* $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} A^*$ *such that* $\sigma = \alpha \circ \beta$. *If* σ *is not recognizable in* $X(\sigma)$, *then* $\sigma \circ \alpha$ *is not fully recognizable. The same statement holds for the recognizability for aperiodic points.*

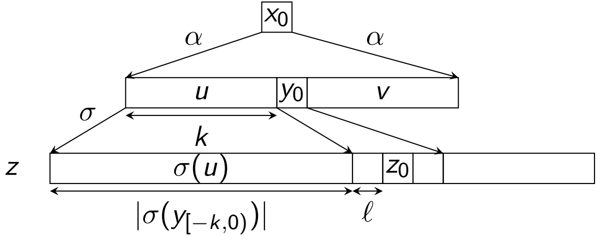
Proof



Proof of the lemma

If σ is not recognizable in $X(\sigma)$ then there exists $z \in X(\sigma)$ with two centered σ-representations $(y, \ell) \neq (y', \ell')$ in $X(\sigma)$. Let (x, k) and (x', k') be centered α-representations in $B^{\mathbb{Z}}$ of y and y' respectively (They exist since (x, k) and (x', k') have σ-representations in $X(\sigma)$).
Then $(x, |\sigma(y_{[-k,0)})| + \ell)$ and $(x', |\sigma(y'_{[-k',0)})| + \ell')$ are centered $\sigma \circ \alpha$-representations of z in $B^{\mathbb{Z}}$.

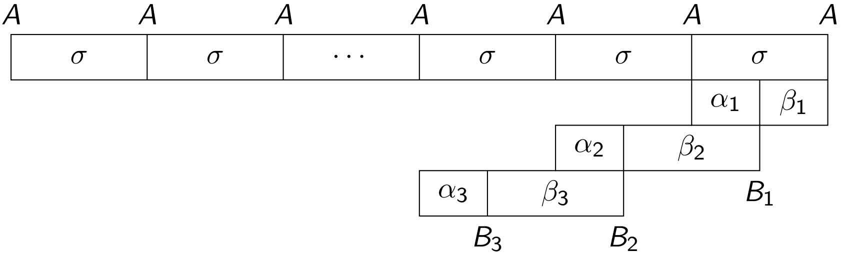
Proof



Proof of the lemma

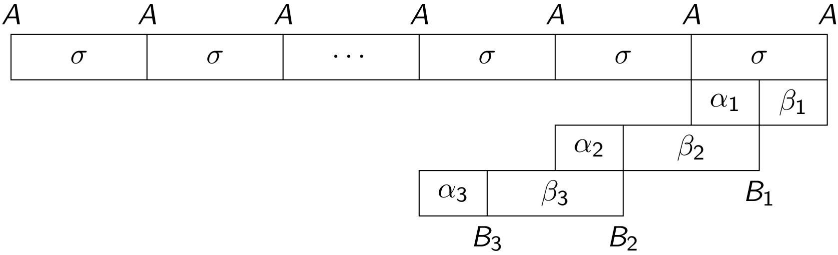
If $(x, |\sigma(y_{[-k,0)})|+\ell) = (x', |\sigma(y'_{[-k',0)})|+\ell')$, then $x = x'$, $u = u'$, $y_0 = y'_0$, $v = v'$. Thus, $k = k'$, $\ell = \ell'$, and $y = y'$.
Further, if z is aperiodic, y also since $z = S^{\ell}(y)$.

Proof



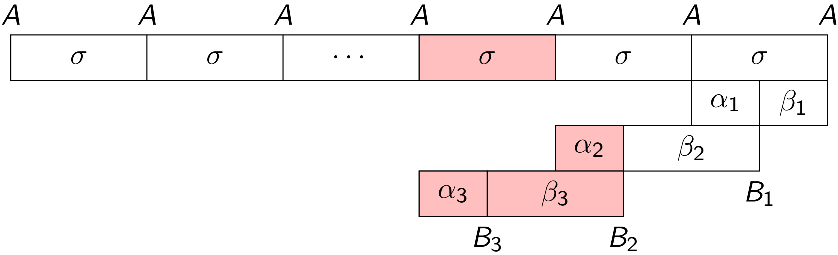
Proof of the theorem

Let $\sigma\colon A^* \to A^*$ be a substitution.
Let us assume that σ is not recognizable in $X(\sigma)$ for aperiodic points.



Proof of the theorem

Let $\sigma\colon A^* \to A^*$ be a substitution.
Let us assume that σ is not recognizable in $X(\sigma)$ for aperiodic points.
We define $\alpha_0\colon A^* \to A^*$ as the identity substitution.
Thus, $\sigma = \sigma \circ \alpha_0$ is not elementary. We decompose it into $\alpha_1 \circ \beta_1$ through B_1 such that $\mathrm{Card}(B_1) < \mathrm{Card}(A)$.
Then, $\sigma \circ \alpha_1$ is not fully recognizable for aperiodic points by the above lemma.
Thus, $\sigma \circ \alpha_1$ is not elementary.



Proof of the theorem

We decompose it into $\sigma \circ \alpha_1 = \alpha_2 \circ \beta_2$ through B_2 such that $\text{Card}(B_2) < \text{Card}(B_1)$.
Again, $\sigma \circ \alpha_2$ is not fully recognizable for aperiodic points and thus not elementary.

Inductively, we define $\sigma \circ \alpha_i = \alpha_{i+1} \circ \beta_{i+1}$ through B_{i+1} such that $\text{Card}(B_{i+1}) < \text{Card}(B_i)$.
We get a contradiction since $\text{Card}(A) < \infty$.