Master 2 Mathematics and Computer Science
Symbolic Dynamics. Exercises

Marie-Pierre Béal

University Gustave Eiffel
Laboratoire d'informatique Gaspard-Monge UMR 8049

Exercises

Exercises on shift spaces

Exercise 1.1

Show that a finite union of shift spaces is a shift space.

Exercises on substitution shifts

Exercise 2.1

Give an example of a substitution $\sigma\colon A^* \to A^*$ be a substitution such that there exists an integer $n \geq 1$ such that $\mathrm{X}(\sigma^n) \neq \mathrm{X}(\sigma)$.

Exercise 2.2

Let $\sigma\colon A^* \to A^*$ be a substitution. Prove that if $\mathrm{X}(\sigma)$ is minimal, then $\mathrm{X}(\sigma^n) = \mathrm{X}(\sigma)$ for every $n \geq 1$.

Exercise 2.3

Let $\sigma\colon A^* \to A^*$ be a substitution. A letter a is erasable if $\sigma^n(a) = \varepsilon$ for some $n \geq 1$. The mortality exponent of an erasable word u, denoted $\mathrm{mex}(u)$ is the least integer n such that $\sigma^n(u) = \varepsilon$. Prove that $\mathrm{mex}(u) \leq \mathrm{Card}(A)$, that is, $\sigma^{\mathrm{Card}(A)}(u) = \varepsilon$.

Exercises on substitution shifts

Exercise 2.4

Let $\sigma\colon A^* \to A^*$ be a substitution. Prove that there are computable integers $i \geq 0$ and $p \geq 1$ depending only on σ such that $\sigma^i(a) = \sigma^{i+p}(a)$ for every non-growing letter $a \in A$.

Exercise 2.5

Let $\sigma\colon A^* \to A^*$ be a substitution. Prove that $\mathcal{L}(\sigma) = \mathcal{B}(\mathrm{X}(\sigma))$ if and only if every letter $a \in A$ is in $\mathcal{B}(\mathrm{X}(\sigma))$.

Exercise 2.6

Let $\sigma\colon A^* \to A^*$ be a primitive substitution, distinct from the identity on a one-letter alphabet. Then,

1. $\mathrm{X}(\sigma^n) = \mathrm{X}(\sigma)$ for all $n \geq 1$.
2. $\mathcal{L}(\sigma) = \mathcal{B}(\mathrm{X}(\sigma))$.
3. For every $w \in \mathcal{L}(\sigma)$ and every $n \geq 1$, there is $u \in \mathcal{L}(\sigma)$ such that w occurs in $\sigma^n(u)$.

Exercises on substitution shifts

Exercise 2.7

Let $\sigma \colon A^* \to A^*$ be the substitution
$a \mapsto bcd, b \mapsto be, c \mapsto fg, d \mapsto hd, e, f, g, h \mapsto \varepsilon$.
Compute the graph $G(\sigma)$ and find the erasable letters, the growing letters, the non-growing letters.

Exercise 2.8

Let $\sigma \colon A^* \to A^*$ be a substitution. Show that a letter a is erasable if all paths going out of a end only in trivial strongly connected components of $G(\sigma)$.
Derive from this an algorithm in linear time in the size of $G(\sigma)$to compute the set of erasable letters.
A trivial strongly connected component is a strongly connected component of a graph that has no edge starting and ending in this component.

Exercise 2.9, more difficult

Show that the set of blocks of the Thue-Morse shift is overlap-free, that is, does not contain words of the form *cucuc* where u is a word and c is a letter.
Hint: show by induction on the size of u that we cannot have both u and cuc in $\sigma(A^*)$ for any letter $c \in A$.

Exercise 2.10

Show that if $\sigma : A^* \to A^*$ has constant length k (i.e. $|\sigma(a)| = k$ for each letter a), the dominant eigenvalue of the composition matrix $M(\sigma)$ is equal to k.

Exercise 2.11

Compute the set of return words to 0 in $X(\sigma)$ with
$\sigma: 0 \mapsto 0012, 1 \mapsto 12, 2 \mapsto 012$.

Exercise 3.1

Show that an elementary substitution is non-erasing.

Exercise 3.2

Show that the substitution $\sigma\colon a \mapsto ab, b \mapsto bc, c \mapsto ca$ is not fully recognizable.

Exercise 3.3

Show that the substitution $\sigma\colon a \mapsto ab, b \mapsto bc, c \mapsto abc$ is fully recognizable.