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Funded by the European Union

erc European Research Council
Established by the European Commission

Words with low discrepancy function and dynamical systems

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The chairperson assignment problem

- We are given k states which form a union.
- Every year a union chairperson has to be selected.
- At any time the accumulated number of chairpersons from each state has to be proportional to its weight.

How to get in an effective way a fair assignment?

From assignments to symbolic discrepancy

Take a sequence $u = (u_n)_n \in \{1, \cdots, d\}^{\mathbb{N}}$.

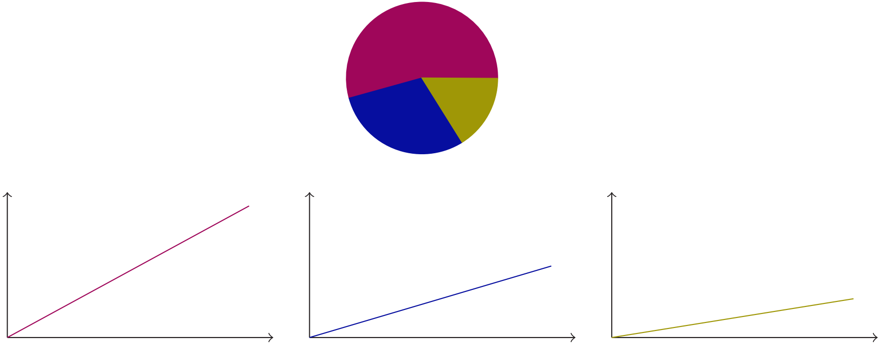
The frequency α_a of the letter a in u is defined as the following limit, if it exists

$$\alpha_a = \lim_{n \to \infty} \frac{1}{n} \mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\}$$

The discrepancy of $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_a \sup_{n \in \mathbb{N}} |\mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a|$$

Discrepancy



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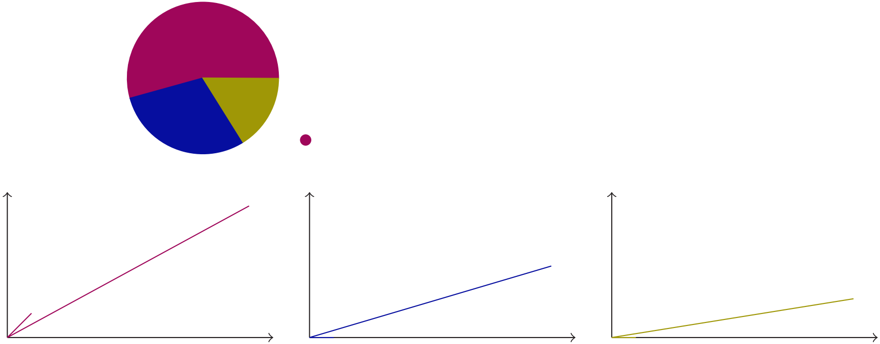

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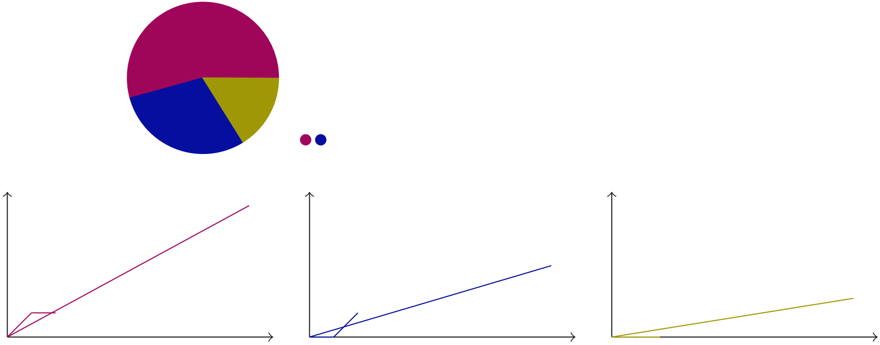

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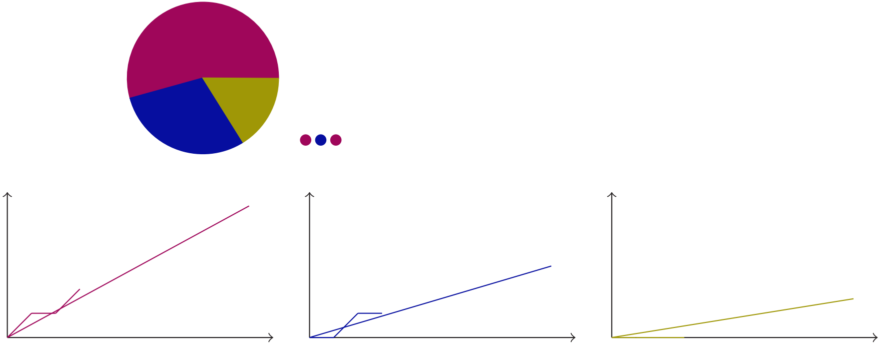

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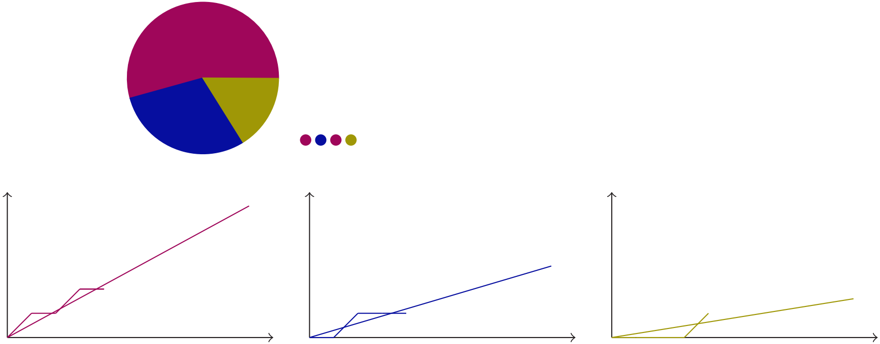

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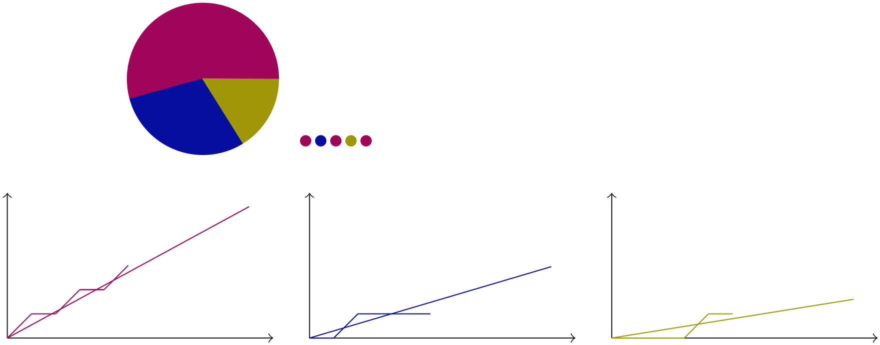

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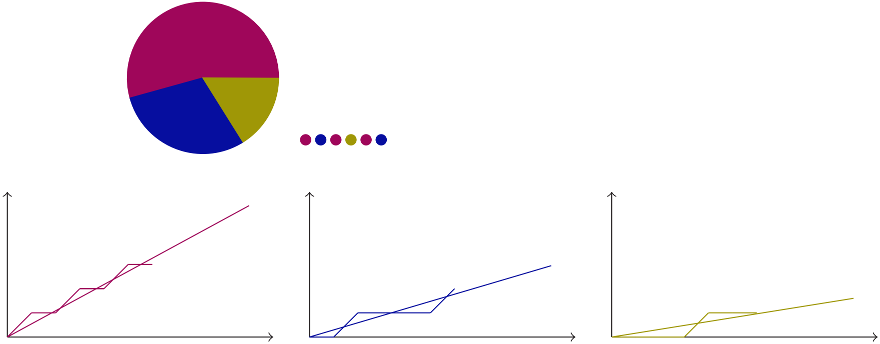

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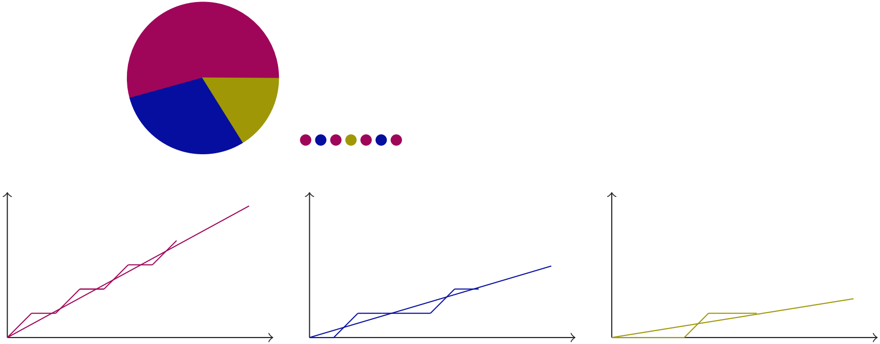

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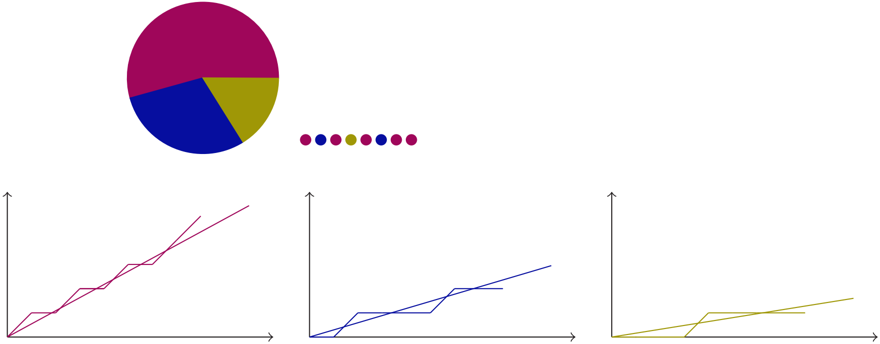

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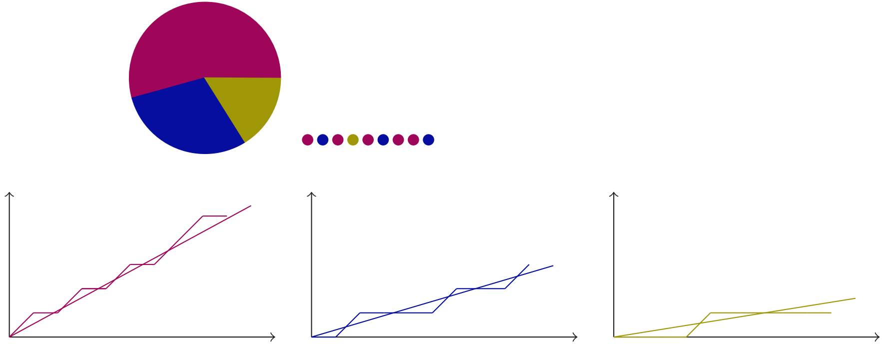

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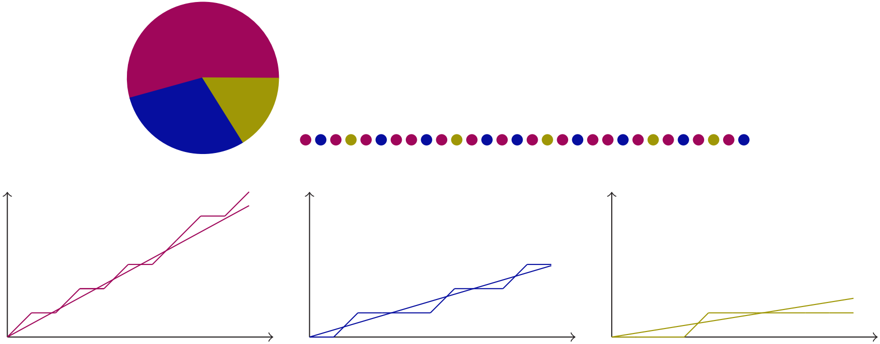

Given a finite alphabet $\mathcal{A}$, and a vector $\boldsymbol{\alpha}$ of frequencies for the letters of $\mathcal{A}$, the aim is to construct an infinite word over $\mathcal{A}$ in which each letter occurs with its prescribed frequency as evenly as possible.

The discrepancy of the word $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_{a \in \mathcal{A}} \sup_{n \in \mathbb{N}} \left| \mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a \right|,$$

for $\boldsymbol{\alpha} = (\alpha_a)_{a \in \mathcal{A}} \in (0,1)^d$ with $\sum_{a \in \mathcal{A}} \alpha_a = 1$.

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Let $d \geq 2$. We consider

$$D_d = \sup_{\boldsymbol{\alpha}} \inf_{u} \Delta_{\boldsymbol{\alpha}}(u)$$

where

- the supremum is taken over the set of frequency vectors $\boldsymbol{\alpha} = (\alpha_i)_{1 \leq i \leq d}$ in $(0,1)^d$ with $\sum_{i=1}^{d} \alpha_i = 1$
- and the infimum over the set of sequences u with values in an alphabet of cardinality d.

The discrepancy of the word $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_{a \in \mathcal{A}} \sup_{n \in \mathbb{N}} \left| \mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a \right|,$$

for $\boldsymbol{\alpha} = (\alpha_a)_{a \in \mathcal{A}} \in (0,1)^d$ with $\sum_{a \in \mathcal{A}} \alpha_a = 1$.

Theorem [Meijer,Tijdeman] Let d stand for the cardinality of $\mathcal{A}$. Let $d \geq 2$. One has

$$D_d = \sup_{\boldsymbol{\alpha}} \inf_{u} \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{2d-2}.$$

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for $\boldsymbol{\alpha} = (\alpha_a)_{a \in \mathcal{A}} \in (0,1)^d$ with $\sum_{a \in \mathcal{A}} \alpha_a = 1$.

- R. Tijdeman has given an algorithmic way, given a frequency vector $\boldsymbol{\alpha} = (\alpha_a)_{a \in \mathcal{A}}$, to construct a sequence u with

$$\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}.$$

- Remark If $\boldsymbol{\alpha}$ has rationally independent coordinates, then by [Schneider]

$$1 - 1/d \leq \Delta_{\boldsymbol{\alpha}}(u).$$

Apportionment problems

See [M.L. Balinski and H.P. Young] for their quota-method for the (discrete) apportionment problem.
This problem, which has its origins in the problem of seat assignments to the house of representatives in the United States, consists in allocating seats in a proportional way.

See also [Altman-Gaujal-Hordijk,Brauner-Crama,Brauner-Jost,Coppersmith-Nowicki-Paleologo-Tresser-Wu,Li,etc.]

How to define an infinite ordered word?

- It has few ‘subwords’
- It is well distributed
- It codes a simple dynamical system

A natural measure of order: factor complexity

Let $\mathcal{A}$ be a finite alphabet and let $u \in \mathcal{A}^{\mathbb{N}}$ be an infinite word

$$u = abaababaabaababaababaab\cdots$$

$$u = abaababaab\underbrace{aa}babaababaab\cdots$$

aa is a factor, bb is not a factor

The factor complexity $p_u(n)$ counts the number of factors of length n

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The factor complexity $p_u(n)$ counts the number of factors of length n

A word u has linear complexity function if there exists a constant $C' > 0$ such that the number of factors of u of length n is smaller than $C' \cdot n$, for every positive integer n

Topological entropy $h(u) = \lim_{n \to +\infty} \frac{\log_d(p_u(n))}{n}$ where d is the cardinality of the alphabet

Outline

- R. Tijdeman has given an algorithmic way to construct fairly distributed sequences u with $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$
- When $d = 2$, $D_2 = 1/2 \rightsquigarrow$ Sturmian sequences
- We revisit Tijdeman's construction in dynamical terms
- We provide constructions of fairly distributed sequences

The two-letter case

The sequences having the smallest discrepancy on a two-letter alphabet are Sturmian sequences.

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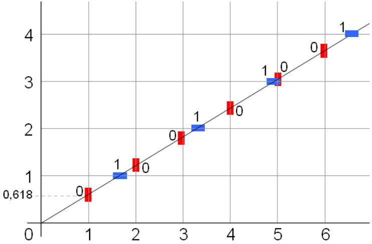
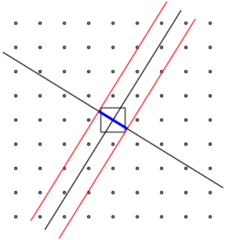
Sturmian sequences are codings of discrete lines.





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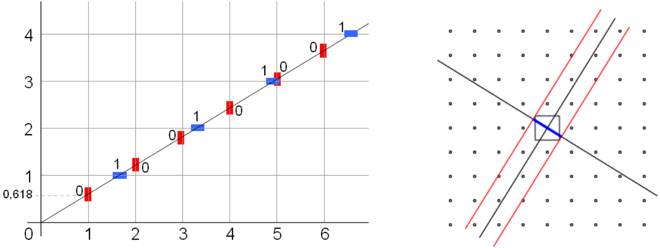
Sturmian sequences are codings of trajectories of dynamical systems.



A trajectory for a discrete-time dynamical system

We consider orbits/trajectories of points of X under the action of the map $T : X \to X$

$$\{T^n x \mid n \in \mathbb{N}\}$$

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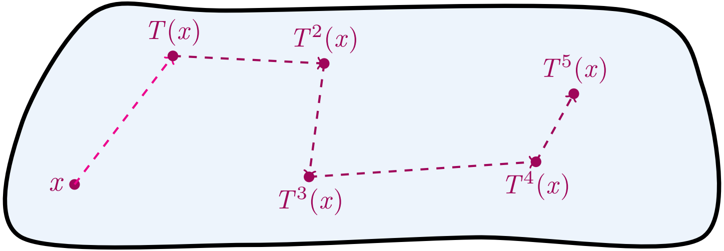

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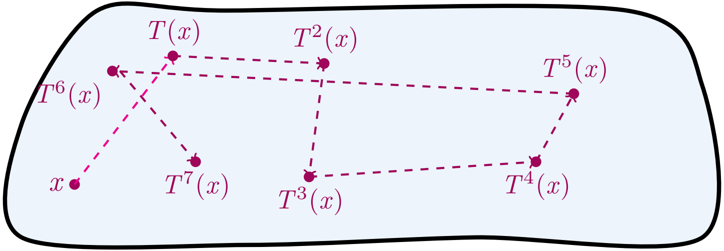
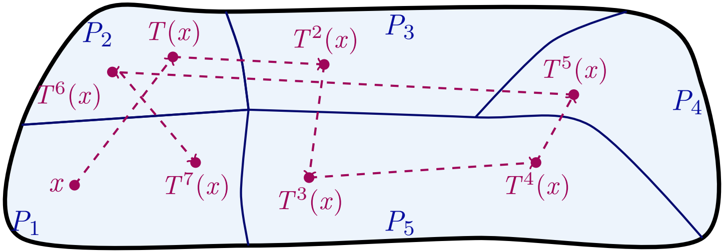


And a coding of a trajectory



The coding works as follows

$$u_n = i \text{ if and only if } T^n(x) \in P_i$$

$$u = (u_n)_n = 12355421 \cdots$$

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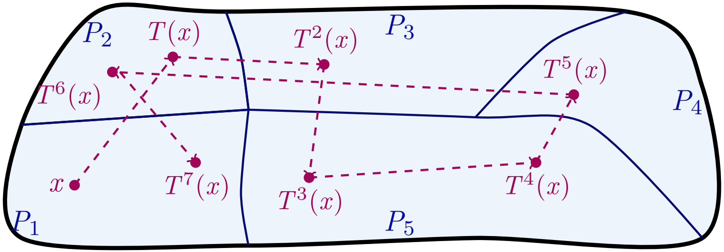

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$$u_n = i \text{ if and only if } T^n(x) \in P_i$$

$$u = (u_n)_n = 12355421\cdots$$

$$x \mapsto Tx, \;\; u \mapsto 2355421\cdots$$

Symbolic dynamics

- The shift T acts on $\mathcal{A}^{\mathbb{Z}}$ as $T((u_n)_n) = (u_{n+1})_n$
- A subshift (X, T) is a closed shift-invariant subset of $\mathcal{A}^{\mathbb{Z}}$
- Cylinders $[v] = \{u \in X,\ u_0 \cdots u_{|v|-1} = v\} \rightsquigarrow$ Intervals
- Factors/Subwords

$$u = abaababaab\underbrace{aa}babaababaab\cdots$$

aa is a factor, bb is not a factor

- The factor complexity $p_X(n)$ counts the number of factors of length n

Symbolic models for circle rotations

The sequences having the smallest discrepancy on a two-letter alphabet are Sturmian sequences

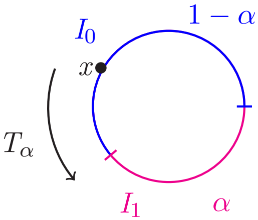
Consider the translation $(\mathbb{T}, T_\alpha)$ where $T_\alpha \colon x \mapsto x + \alpha \bmod 1$ and the coding map

$$\nu \colon [0,1) \to \{0,1\}, \quad \nu(x) = 0 \quad \text{if } x \in I_0, \quad \nu(x) = 1 \quad \text{if } x \in I_1$$

where

$$I_0 = [0, 1-\alpha), \ I_1 = [1-\alpha, 1)$$

The trajectory of x for T_α is coded by $u \in \{0,1\}^{\mathbb{Z}}$ with $u_n = \nu(T_\alpha^n(x))$ for all n



A natural measure of order: factor complexity

What kind of information can the dynamical viewpoint offer here?

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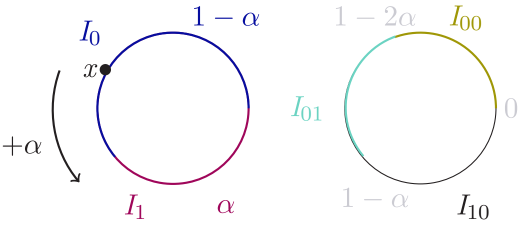
$$u = 0100101001001010010100 1\cdots$$

Does the word 00 occur in the sequence? Does it have a frequency? Does it have bounded discrepancy?



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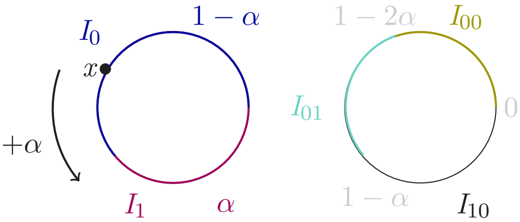
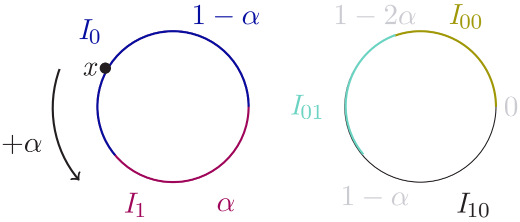


The factors of length n of u are in one-to-one correspondence with the $n+1$ intervals of $\mathbb{T}$ whose end-points are given by

$$-k\alpha \bmod 1 \qquad \text{for } 0 \leq k \leq n$$

By uniform distribution of $(k\alpha)_k$ modulo 1, the frequency of a factor w of a Sturmian sequence is equal to the length of I_w

Bounded remainder sets



Bounded remainder set A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\text{Card}\{0 \leq n \leq N; T_\alpha^n(0) \in X\} - N\mu(X)| \leq C$$

[Kesten'66] Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

Letters and even all the factors of Sturmian sequences have bounded discrepancy

Discrepancy for Kronecker sequences

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in [0,1]^d$ with $1, \alpha_1, \cdots, \alpha_d$ $\mathbb{Q}$-linearly independent. Consider the Kronecker sequence in $[0,1]^d$

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

associated with the translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d, \; x \mapsto x + \boldsymbol{\alpha}$$

One has

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\}) = T_{\boldsymbol{\alpha}}^n(0)$$

Discrepancy for Kronecker sequences

Consider the minimal translation over $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d, \; x \mapsto x + \boldsymbol{\alpha} \mod 1, \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$$

Discrepancy Global property

$$\Delta_N(\boldsymbol{\alpha}) = \sup_{B \text{ box}} \left|\mathrm{Card}\,\{0 \leq n < N;\, T_{\boldsymbol{\alpha}}^n(0) \in B\} - N \cdot \mu(B)\right|$$

[Khintchine, Beck] $\Delta_N(\boldsymbol{\alpha})$ is a.e. between

$$(\log N)^d \log\log N \quad \text{and} \quad (\log N)^d (\log\log N)^{1+\varepsilon}$$

Bounded remainder set Local property A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N;\, T_{\boldsymbol{\alpha}}^n(0) \in X\} - N\mu(X)| \leq C$$

Bounded remainder sets for toral translations

Bounded remainder set A measurable set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N;\, T_{\boldsymbol{\alpha}}^{n}(0) \in X\} - N\mu(X)| \leq C$$

[Kesten'66] $d = 1$ Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

[Grepstad-Lev'15, Haynes-Kelly-Koivusalo'17] Any parallelotope in $\mathbb{R}^d$ spanned by vectors $v_1, \cdots, v_d$ belonging to $\mathbb{Z}\boldsymbol{\alpha} + \mathbb{Z}^d$ is a bounded remainder set for the minimal translation $T_{\boldsymbol{\alpha}}$

$$T_{\boldsymbol{\alpha}} \colon \mathbb{T}^d \to \mathbb{T}^d,\ x \mapsto x + \boldsymbol{\alpha} \mod 1, \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$$

The ubiquitous Fibonacci word

Take the golden ratio $\alpha = \frac{\sqrt{5}+1}{2}$ and the dynamical system

$$x \mapsto x + \alpha \text{ modulo } 1$$

$$\alpha^2 = \alpha + 1 \rightsquigarrow \text{self-similarity}$$

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$$\alpha^2 = \alpha + 1 \rightsquigarrow \text{self-similarity} \rightsquigarrow \text{substitution}$$

The Fibonacci substitution

$$\sigma(u) = u \text{ with } \sigma : 0 \mapsto 01,\ 1 \mapsto 0$$

$$u = \sigma^\omega(1) = 010010100100101 \cdots$$

Theorem The symbolic dynamical system (X_σ, T) is isomorphic to the geometric dynamical system $(\mathbb{T}, T_{\frac{1+\sqrt{5}}{2}})$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

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Zeckendorf numeration

$$n = \sum_{i=1}^{k} \varepsilon F_i, \varepsilon_i \in \{0, 1\},\ 11 \nexists$$

Fair assignments in general dimension

The best assignments for $d = 2$ code the simplest (discrete-time) dynamical systems.

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And now for $d \geq 3$?

- Given a frequency vector $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in [0,1]^d$ such that $\sum_{i=1}^d \alpha_i = 1$, R.Tijdeman ('80) has given an algorithmic way to construct a sequence u with $\Delta_{\boldsymbol{\alpha}}(u) \leq 1 - \frac{1}{2d-2}$.

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Theorem [B.-Carton-Chevallier-Steiner-Yassawi] Let u be a Tijdeman sequence with a frequency vector $\boldsymbol{\alpha}$ which has rationally independent coordinates. Then, the sequence u has factor complexity of order n^{d-1}.

The sequence u is a symbolic coding of a translation $T_{\boldsymbol{\alpha}}$ via a partition of a fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of polytopes such that $T_{\boldsymbol{\alpha}}$ is a translation by a vector on each of the polytopes.

Dynamical systems and Tijdeman sequences

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Dynamical systems and Tijdeman sequences

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Consider the minimal translation $T_{\boldsymbol{\alpha}}$

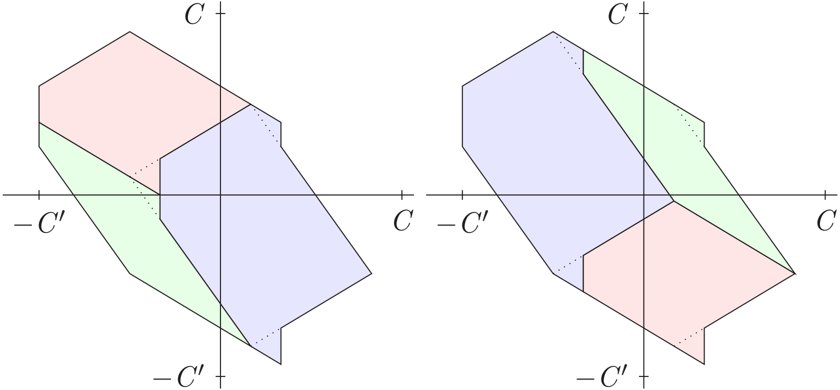
$$T_{\boldsymbol{\alpha}} : \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}, \quad \mathbf{x} \mapsto \mathbf{x} + (\alpha_1, \ldots, \alpha_{d-1}) \pmod{\mathbb{Z}^{d-1}}.$$

Theorem [B.-Carton-Chevallier-Steiner-Yassawi] Let u be a Tijdeman sequence with $\boldsymbol{\alpha} = (\alpha_i)_{1 \leq i \leq d}$ having rationally independent coordinates.

- The sequence u has factor complexity of order n^{d-1}.
- The sequence u is a symbolic coding of $T_{\boldsymbol{\alpha}}$ via a partition of a fundamental domain of $\mathbb{T}^{d-1}$ into d finite unions of polytopes such that $T_{\boldsymbol{\alpha}}$ is a translation by a vector on each of the polytopes.
- The sequence u generates a minimal and uniquely ergodic subshift which has discrete spectrum.

A fundamental domain by polygons

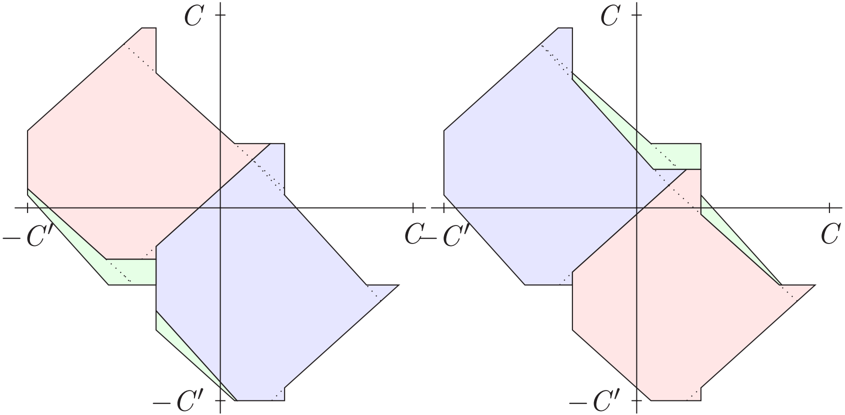
Take $d = 3$, $\boldsymbol{\alpha} \approx (.5, .3, .2)$, $C = C' = 3/4$



Tijdeman sequences code orbits of the corresponding exchange of domains.

This yields a factor complexity of order $n^{d-1} = n^2$

Take $d = 3$, $\boldsymbol{\alpha} \approx (.5, .45, .05)$, $C = C' = 3/4$. The atoms of the partition are unions of polygons.



What does “order” mean for subshifts?

A subshift (X, T) with $X \subset \mathcal{A}^{\mathbb{Z}}$ is simple if

- it has few factors $p_X(n) \leq Cn$ for all n
- it has bounded discrepancy for letters and factors
- it codes a simple dynamical system (a group translation)

What are the relations between these notions of order?

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Theorem [D. Creutz, R. Pavlov] If $\limsup p_X(n)/n < 3/2$, then X is measurably isomorphic to a group translation

Fairly distributed shifts

How to construct minimal shifts X over the alphabet $\{1, 2, \cdots, d\}$ satisfying the following conditions

- the letter frequencies in X are given by $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$
- X has bounded discrepancy for all its factors
- X has linear factor complexity

Fairly distributed shifts

How to construct minimal shifts X over the alphabet $\{1, 2, \cdots, d\}$ satisfying the following conditions

- the letter frequencies in X are given by $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$
- X has bounded discrepancy for all its factors
- X has linear factor complexity
- X is a symbolic coding of a toral translation

Let us start from the dynamical system given by the translation

$$T_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha} \mod 1$$

How to find a good partition?

How to produce symbolic codings for translations

How to produce fair assignments/ fairly distributed sequences/symbolic codings of $T_{\boldsymbol{\alpha}}$ for a given vector of letter frequencies $\boldsymbol{\alpha}$?

- We apply a multidimensional continued fraction algorithm that generates nonnegative matrices

$$\boldsymbol{\alpha} \mapsto (M_n)_n \text{ with } \quad \boldsymbol{\alpha} \in \bigcap_n M_1 \cdots M_n \mathbb{R}_+^d$$

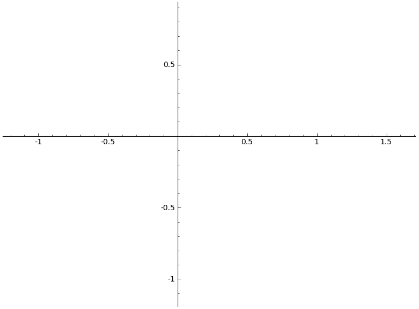
- that generates in turn a sequence of substitutions $\boldsymbol{\alpha} \mapsto (M_n)_n \mapsto \boldsymbol{\sigma} = (\sigma_n)_n$

- and thus sequences $u = \lim \sigma_0 \cdots \sigma_n(a) \rightsquigarrow (X_{\boldsymbol{\sigma}}, T)$ (S-adic formalism)

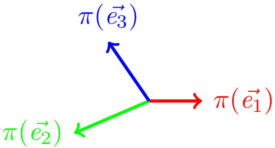
Rauzy fractal and the Tribonacci substitution

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1 \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\sigma^{\omega}(1) = 1213121121312121312112 13\cdots$$



π projection along the expanding eigenline onto the contracting plane of the incidence matrix of



M_σ

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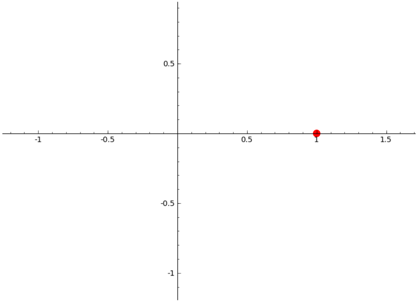

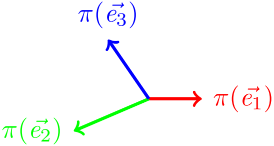

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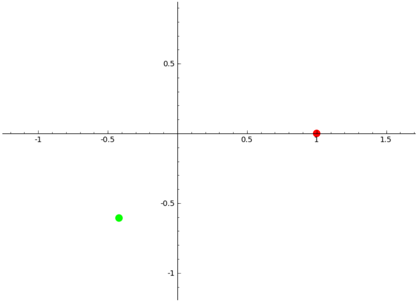

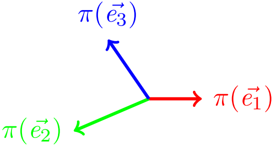

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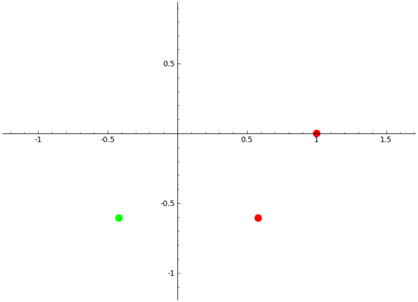

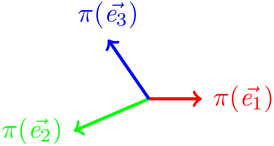

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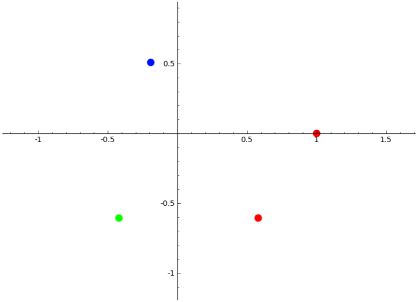

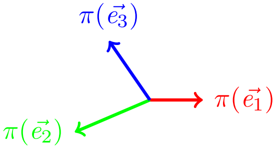

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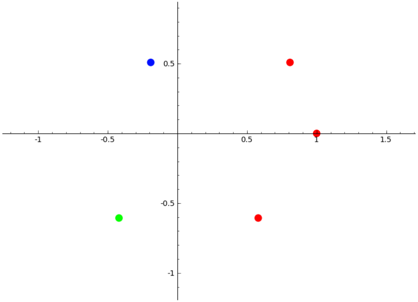

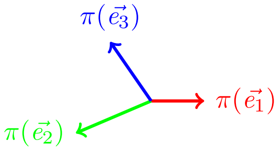

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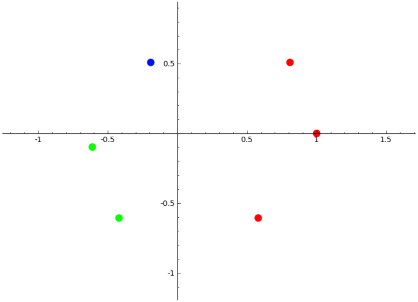

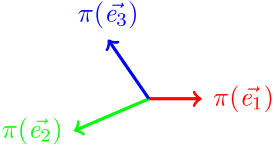

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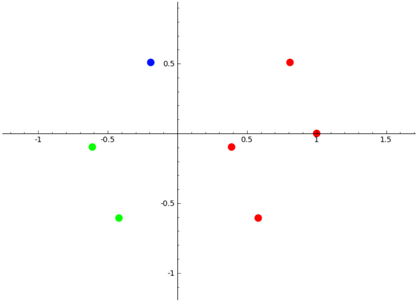

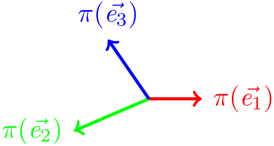

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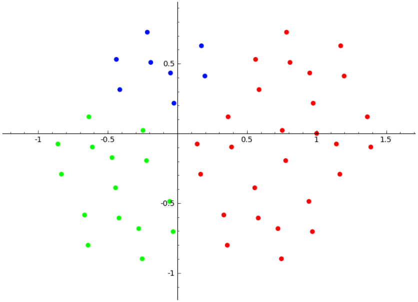

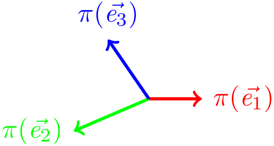

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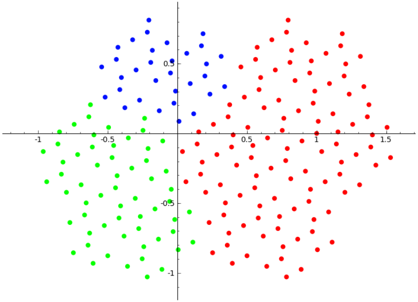

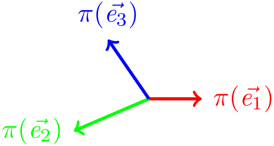

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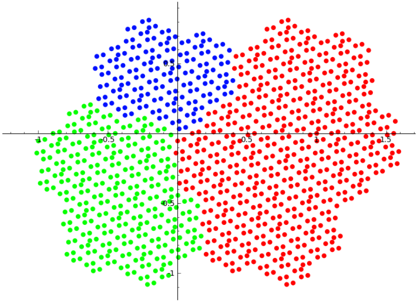

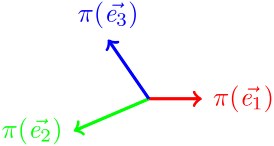

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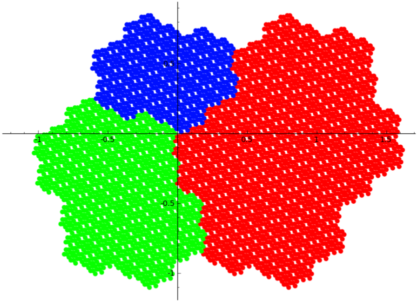

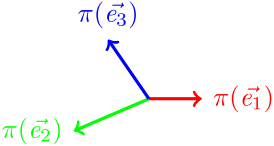

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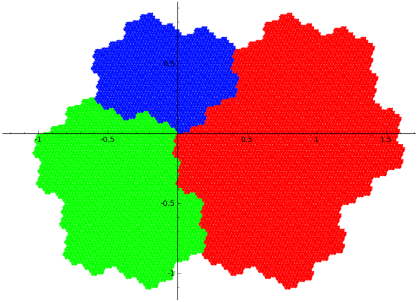
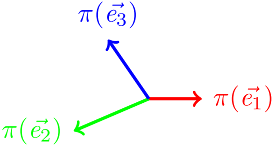




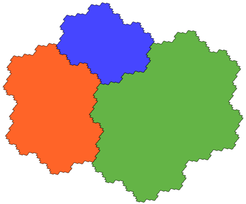
Pisot numbers, codings and fractals

$$X^3 = X^2 + X + 1$$

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation T_β on the two-dimensional torus $\mathbb{T}^2$

$$T_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



Beyond the Pisot conjecture

Classical exponentially convergent multidimensional continued fraction algorithms generate faithful symbolic codings for translations on the torus.

Take your favourite algorithm A.

Theorem [B.-Steiner-Thuswaldner, Pytheas Fogg-Noûs]
For almost every $\boldsymbol{\alpha} \in [0,1]^d$, the translation $T_{\boldsymbol{\alpha}} : \mathbf{x} \mapsto \mathbf{x} + \boldsymbol{\alpha}$ on the torus $\mathbb{T}^d$ admits a symbolic model: the S-adic system provided by the multidimensional continued fraction algorithm A applied to $\boldsymbol{\alpha}$ is isomorphic in measure to $T_{\boldsymbol{\alpha}}$. Moreover, factors have bounded discrepancy.

And now?

The discrepancy is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_a \sup_{n\in\mathbb{N}} \left|\mathrm{Card}\{k, 0 \leq k \leq n-1, u_k = a\} - n\alpha_a\right|$$

One has

$$\sup_{\boldsymbol{\alpha}} \inf_u \Delta_{\boldsymbol{\alpha}}(u) = 1 - \frac{1}{2d-2}$$

Now, given $\boldsymbol{\alpha}$, what about

$$\inf_u \Delta_{\boldsymbol{\alpha}}(u)?$$