Numeration systems

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Outline

Binary numerations

Gaussian integers

2-adic integers

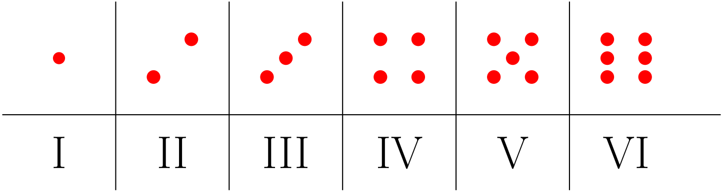
Zeckendorf numeration

Z-adic integers

Numeration systems

A numeration system (or numeration for short) is a way to *represent* numbers.

Number $\longleftrightarrow$ Finite sequence of digits



In this talk, we focus on integers because this is simpler but there are also numerations to represent real numbers.

Outline

Binary numerations

Gaussian integers

2-adic integers

Zeckendorf numeration

Z-adic integers

Binary numeration aka base 2 numeration

Each non-negative integer $n \in \mathbb{N}$ is a sum of powers of 2.

$$n = \sum_{\ell=0}^{k} d_\ell 2^\ell \qquad \text{where} \qquad d_\ell \in \{0, 1\}$$

Number	Expansion
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
$\vdots$	$\vdots$

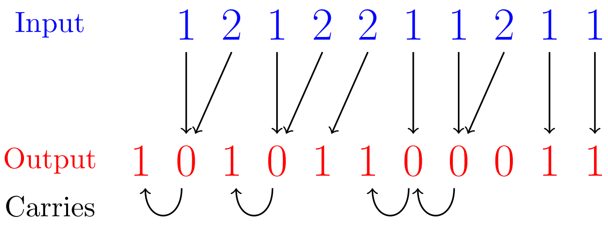
Binary numeration with digits 1 and 2

Each non-negative integer $n \in \mathbb{N}$ is a sum of powers of 2.

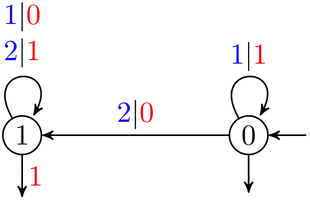
$$n = \sum_{\ell=0}^{k} d_\ell 2^\ell \qquad \text{where} \qquad d_\ell \in \{1, 2\}$$

Number	Expansion
0	ε
1	1
2	2
3	11
4	12
5	21
6	22
7	111
8	112
9	121
$\vdots$	$\vdots$

From digits in $\{1, 2\}$ to digits in $\{0, 1\}$



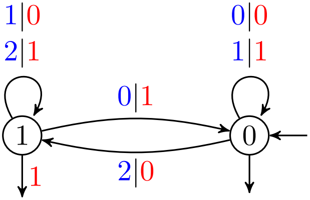
Transducer realizing
the transformation
from right to left



Addition: from digits in $\{0, 1, 2\}$ to digits in $\{0, 1\}$

$$
\begin{array}{lccccccccccc}
 & & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
+ & & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
\text{Input} & & 1 & 2 & 1 & 2 & 2 & 1 & 0 & 2 & 1 & 0 \\
\text{Output} & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
\text{Carries} & & & & & & & & & & &
\end{array}
$$

Transducer realizing
the transformation
from right to left



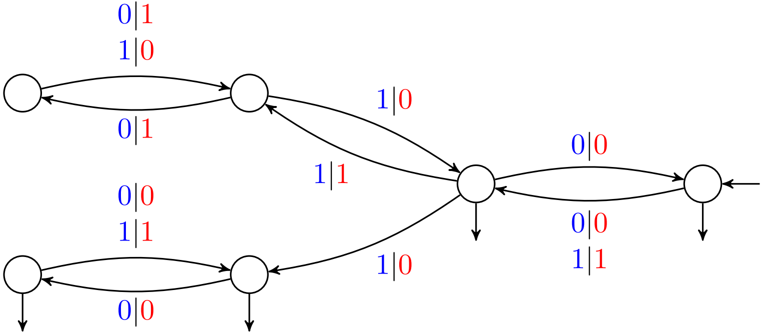
Binary numeration with powers of -2

Each integer $n \in \mathbb{Z}$ is a sum of powers of -2.

$$n = \sum_{\ell=0}^{k} d_\ell (-2)^\ell \qquad \text{where} \qquad d_\ell \in \{0, 1\}$$

Number	Expansion	Number	Expansion
0	0	-10	1010
1	1	-9	1011
-2	10	-4	1100
-1	11	-3	1101
4	100	-6	1110
5	101	-5	1111
2	110	16	10000
3	111	17	10001
-8	1000	14	10010
-7	1001	15	10011
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Absolute value from powers of -2 to powers of 2



Transducer realizing the transformation from right to left

Outline

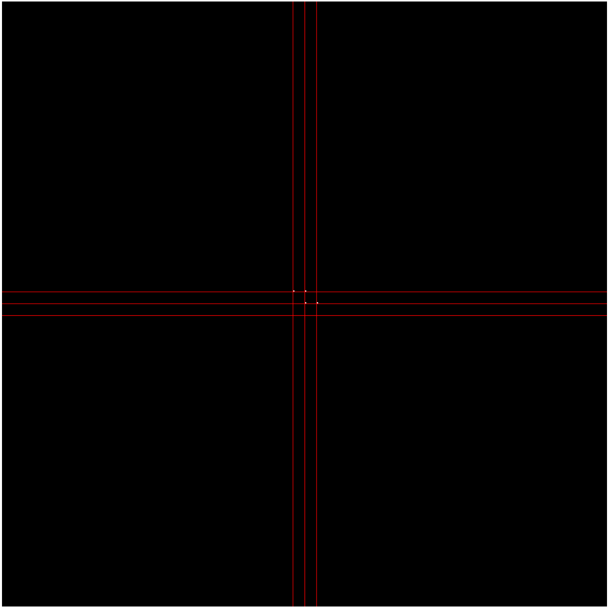
Gaussian integers

Each Gaussian integer $a + bi \in \mathbb{Z}[i]$ is a sum of powers of $-1 + i$.

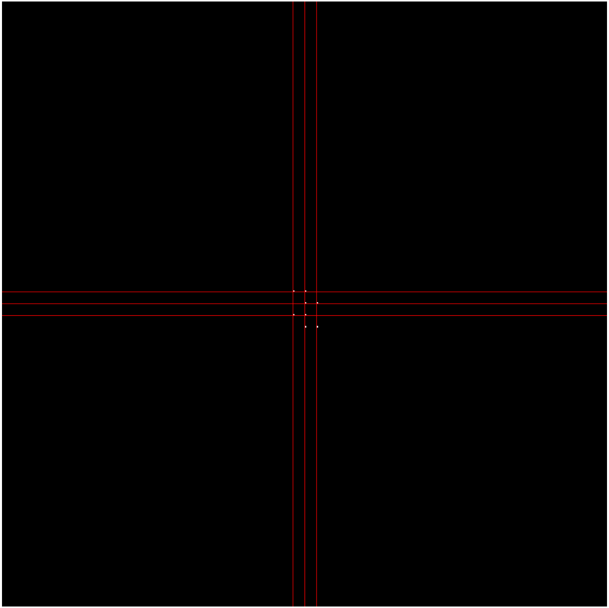
$$a + bi = \sum_{\ell=0}^{k} d_\ell (-1 + i)^\ell \qquad \text{where} \qquad d_\ell \in \{0, 1\}$$

Number	Expansion	Number	Expansion
0	0	$1 + 3i$	1010
1	1	$2 + 3i$	1011
$-1 + i$	10	2	1100
i	11	3	1101
$-2i$	100	$1 + i$	1110
$1 - 2i$	101	$2 + i$	1111
$-1 - i$	110	-4	10000
$-i$	111	-3	10001
$2 + 2i$	1000	$-5 + i$	10010
$3 + 2i$	1001	$-4 + i$	10011
$\vdots$	$\vdots$	$\vdots$	$\vdots$

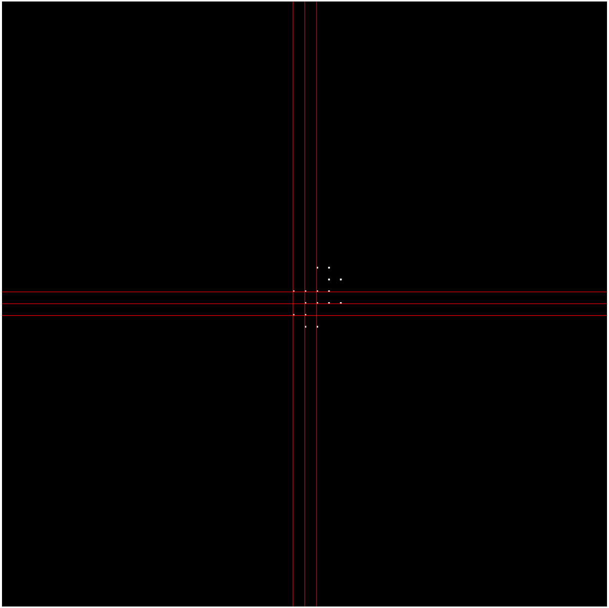
Gaussian integers with an expansion of length 2



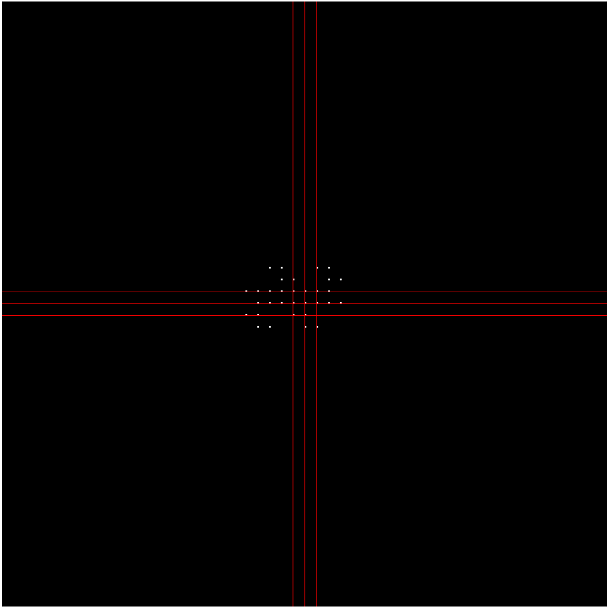
Gaussian integers with an expansion of length 3



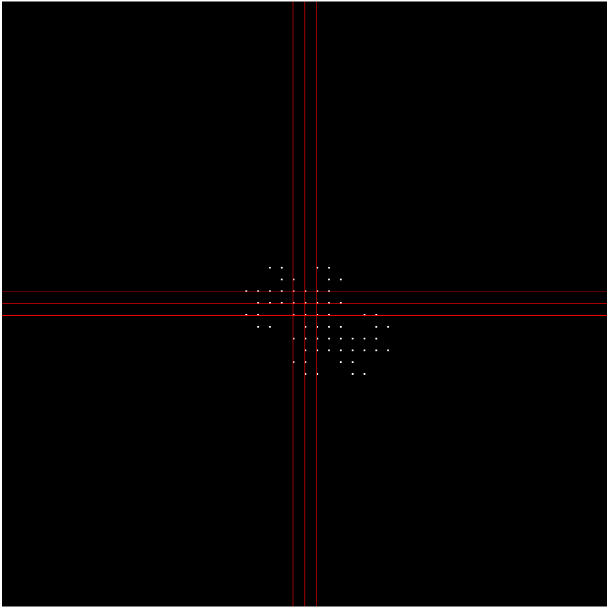
Gaussian integers with an expansion of length 4



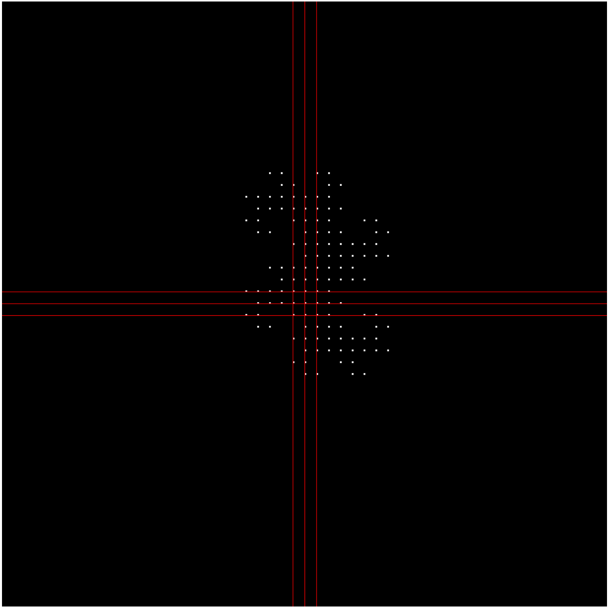
Gaussian integers with an expansion of length 5



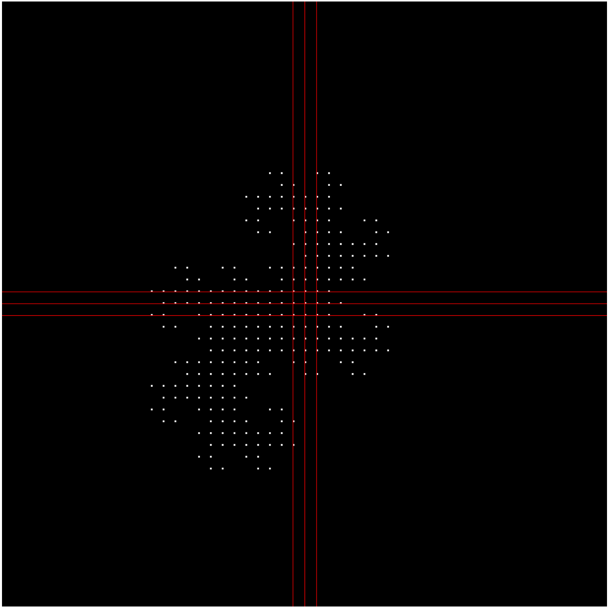
Gaussian integers with an expansion of length 6



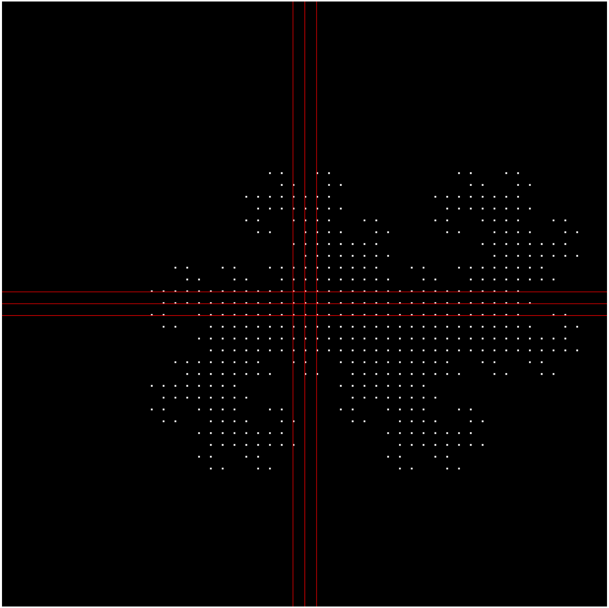
Gaussian integers with an expansion of length 7



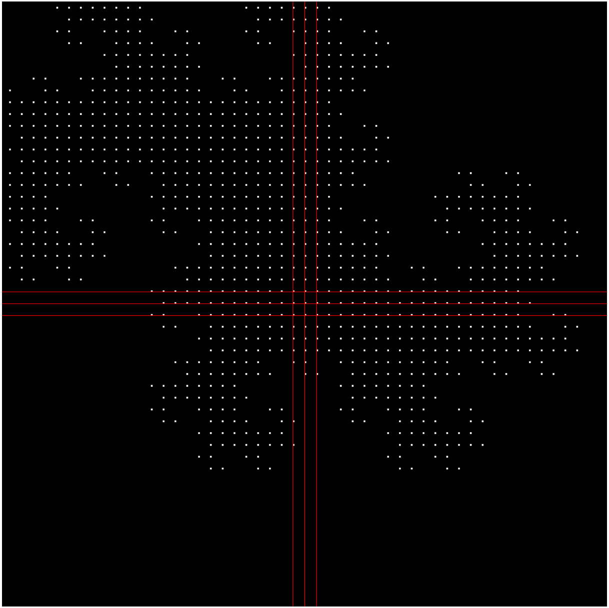
Gaussian integers with an expansion of length 8



Gaussian integers with an expansion of length 9



Gaussian integers with an expansion of length 10



Outline

2-adic integers

Suppose

$$
\begin{aligned}
m &= \sum_{\ell \geqslant 0} m_\ell 2^\ell \quad \text{and} \quad n = \sum_{\ell \geqslant 0} n_\ell 2^\ell && \text{where} \quad m_\ell, n_\ell \in \{0,1\} \\
m + n &= \sum_{\ell \geqslant 0} p_\ell 2^\ell && \text{where} \quad p_\ell \in \{0,1\} \\
mn &= \sum_{\ell \geqslant 0} q_\ell 2^\ell && \text{where} \quad q_\ell \in \{0,1\}
\end{aligned}
$$

Each p_ℓ and each q_ℓ only depend on $p_0, \ldots, p_\ell$ and $q_0, \ldots, q_\ell$, and not on $p_{\ell+1}, p_{\ell+2}, \ldots$ and $q_{\ell+1}, q_{\ell+2}, \ldots$.
Therefore, we call have integers with infinitely many digits like

$$\cdots 001010010 \qquad \text{or} \qquad \cdots 01010101$$

and still continue to add and multiply them.

2-adic integers: mathematical viewpoint

Consider the (ring) morphism $\pi_n : \mathbb{Z}/2^{n+1}\mathbb{Z} \to \mathbb{Z}/2^n\mathbb{Z}$. Consider the product $\prod_{n\geqslant 1} \mathbb{Z}/2^n\mathbb{Z}$ and its subset $\mathbb{Z}_2$ defined by

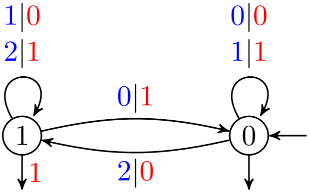
$$\mathbb{Z}_2 = \{(r_n)_{n\geqslant 1} : r_n \in \mathbb{Z}/2^n\mathbb{Z} \text{ and } \pi_n(r_{n+1}) = r_n\}\,.$$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathbb{Z}/32\mathbb{Z} & \xrightarrow{\pi_4} & \mathbb{Z}/16\mathbb{Z} & \xrightarrow{\pi_3} & \mathbb{Z}/8\mathbb{Z} & \xrightarrow{\pi_2} & \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\pi_1} & \mathbb{Z}/2\mathbb{Z} \\
 & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
\cdots & \longrightarrow & r_5 & \xmapsto{\pi_4} & r_4 & \xmapsto{\pi_3} & r_3 & \xmapsto{\pi_2} & r_2 & \xmapsto{\pi_1} & r_1 \\
\cdots & \longrightarrow & 10010 & \xmapsto{\pi_4} & 0010 & \xmapsto{\pi_3} & 010 & \xmapsto{\pi_2} & 10 & \xmapsto{\pi_1} & 0
\end{array}$$

$\cdots 111$ is -1

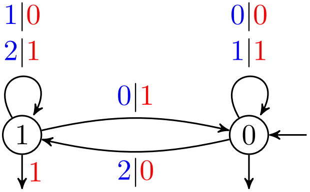
$$\begin{array}{rcccccccccc}
 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
+ & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Input		1	1	1	1	1	1	1	1	1	2
Output	0	0	0	0	0	0	0	0	0	0	0
Carries											



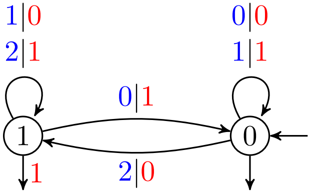
How to obtain -9 ?

$$
\begin{array}{cccccccccccccc}
 & 9 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
+ & ? & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
\hline
 & -1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$



How to obtain -9 ?

$$
\begin{array}{rcccccccccccc}
9 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
-10 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
\cline{2-12}
{\color{blue}-1} & {\color{blue}\cdots} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} & {\color{blue}1} \\
\\
{\color{red}-9} & {\color{red}\cdots} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}1} & {\color{red}0} & {\color{red}1} & {\color{red}1} & {\color{red}1}
\end{array}
$$



In processors

In C:

```c
int main(void) {
  printf("sizeof(int): %ld, -1: %08X\n", sizeof(int), -1);
  return 0;
}
```

```
sizeof(int): 4, -1: FFFFFFFF
```

In Python:

```python
for number in [-2, -1, 0, 1, 2]:
    bytes  = number.to_bytes(length = 2, signed = True)
    string = " ".join(format(byte, "08b") for byte in bytes)
    print(f"{number:3d}: [{string}]")
```

```
 -2: [11111111 11111110]
 -1: [11111111 11111111]
  0: [00000000 00000000]
  1: [00000000 00000001]
  2: [00000000 00000010]
```

Outline


ZECKENDORF BLVD
STEWART AVE
OLD COUNTRY RD

Zeckendorf numeration

Recall the Fibonacci numbers $(F_n)_{n\geqslant 0}$, defined by

$$F_{-2}=0, F_{-1}=1 \quad \text{and} \quad F_n = F_{n-1}+F_{n-2} \text{ for } n \geqslant 0$$

Each non-negative integer $n \in \mathbb{N}$ is a sum of Fibonacci numbers.

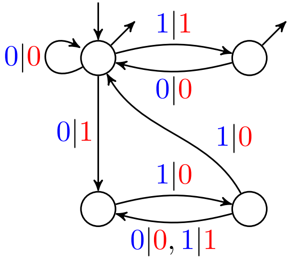
$$n = \sum_{\ell=0}^{k} d_\ell F_\ell \qquad \text{where} \qquad d_\ell \in \{0,1\}$$

Number	Expansion	Number	Expansion
0	0	7	1010
1	1	8	1011
2	10	8	1100
3	11	9	1101
3	100	10	1110
4	101	11	1111
5	110	8	10000
6	111	9	10001
5	1000	10	10010
6	1001	11	10011

Zeckendorf numeration: normalization

Theorem (Zeckendorf)

Each non-negative integer $n \in \mathbb{N}$ has a unique expansion $n = \sum_{\ell=0}^{k} d_\ell F_\ell$ where $d_\ell d_{\ell+1} = 0$ for each $\ell \geqslant 0$ (no 11).



n	55	34	21	13	8	5	3	2	1
61	0	1	0	1	1	1	0	0	1
61	0	1	1	0	0	1	0	0	1
61	1	0	0	0	0	1	0	0	1
61	0	1	0	0	2	2	0	0	1

$$\mathrm{nor}_Z(010111001) = \mathrm{nor}_Z(010111001) = 100001001$$

$$\mathrm{nor}_Z(010022001) = 100001001$$

Outline

Z-adic integers (part 1)

Let $\mathbb{G}$ be the abelian group of bounded sequences of integers with component-wise addition.

$$\mathbb{G} := \mathbb{Z}^{\mathbb{N}} \cap \ell^{\infty} = \{g \in \mathbb{Z}^{\mathbb{N}} : \|g\|_{\infty} < \infty\}.$$

$\nu_Z(w)$ is the number of zeros at the beginning of $\mathrm{nor}_Z(w)$.

$$\nu_Z(w) := \max\{k : \mathrm{nor}_Z(w) \in \{0,1\}^* 0^k\}.$$

Examples

$w \in \mathbb{Z}^*$	value $[w]_Z$	$\mathrm{nor}_Z(w)$	$\nu_Z(w)$
1	1	1	0
21	5	1000	3
121	8	10000	4
1121	13	100000	5
11121	21	1000000	6

Z-adic integers (part 2)

$$\mathbb{H}_Z := \{g \in \mathbb{G} : \lim_{n\to\infty} \nu_Z(g_{[0:n)}) = \infty\}$$

Examples $\cdots 11121 \in \mathbb{H}_Z$ and $\cdots 10101011 \in \mathbb{H}_Z$

$w \in \mathbb{Z}^*$	value $[w]_Z$	$\mathrm{nor}_Z(w)$	$\nu_Z(w)$
11	3	100	2
1011	8	10000	4
101011	21	1000000	6
10101011	55	100000000	8

$$\cdots 11111 \notin \mathbb{H}_Z$$

$w \in \mathbb{Z}^*$	value $[w]_Z$	$\mathrm{nor}_Z(w)$	$\nu_Z(w)$
11	3	100	2
111	6	1001	0
1111	11	10100	2
11111	19	101001	0
111111	32	1010100	2

Z-adic integers (part 3)

Theorem

$\mathbb{H}_Z$ is a subgroup of $\mathbb{G}$ and $\mathbb{Z}_Z$ is defined as $\mathbb{Z}_Z := \mathbb{G}/\mathbb{H}_Z$.

Examples

$\cdots 101010$ and $\cdots 1010101$ are equal to -1 and their difference $\cdots 1\bar{1}1\bar{1}1\bar{1}1 \in \mathbb{H}_Z$ where $\bar{1}$ denotes the digit -1.

$w \in \mathbb{Z}^*$	value $[w]_Z$	$\mathrm{nor}_Z(w)$	$\nu_Z(w)$
1	1	1	0
$\bar{1}1$	-1	-1	0
$1\bar{1}1$	2	10	1
$\bar{1}1\bar{1}1$	-3	-100	2
$1\bar{1}1\bar{1}1$	5	1000	3
$\bar{1}1\bar{1}1\bar{1}1$	-8	-10000	4
$1\bar{1}1\bar{1}1\bar{1}1$	13	100000	5

$\cdots 1111$ is -2 obtained by adding the two expansions of -1.

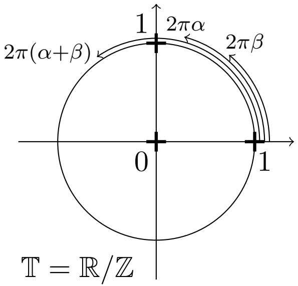
Z-adic integers (part 4)

Theorem

Each class of $\mathbb{Z}_Z = \mathbb{G}/\mathbb{H}_Z$ contains an infinite sequence over $\{0, 1\}$ with no 11.

Theorem

The group $\mathbb{Z}_Z$ is isomorphic to the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.



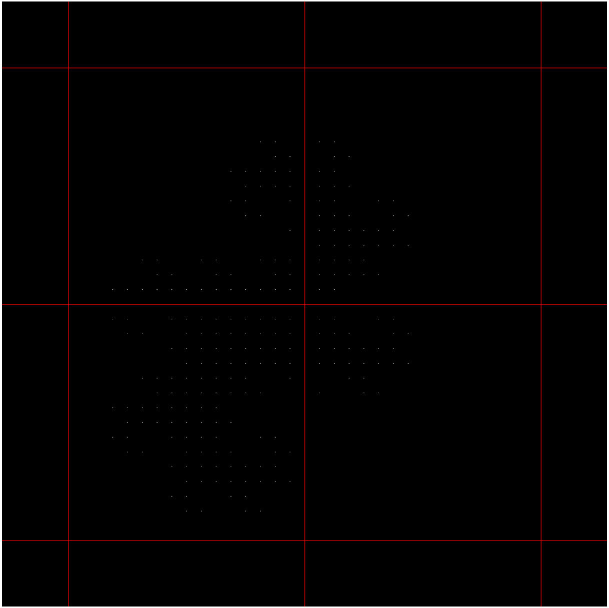
Questions/Open questions

Renormalization of Gaussian integers

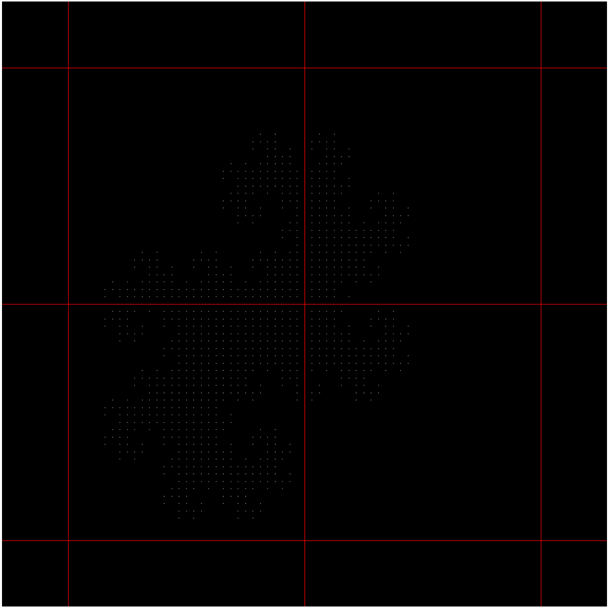
Let us consider sums of the following form

$$\sum_{\ell=0}^{k} d_\ell \left(\frac{-1+i}{2}\right)^{\ell} \qquad \text{where} \qquad d_\ell \in \{0,1\}$$

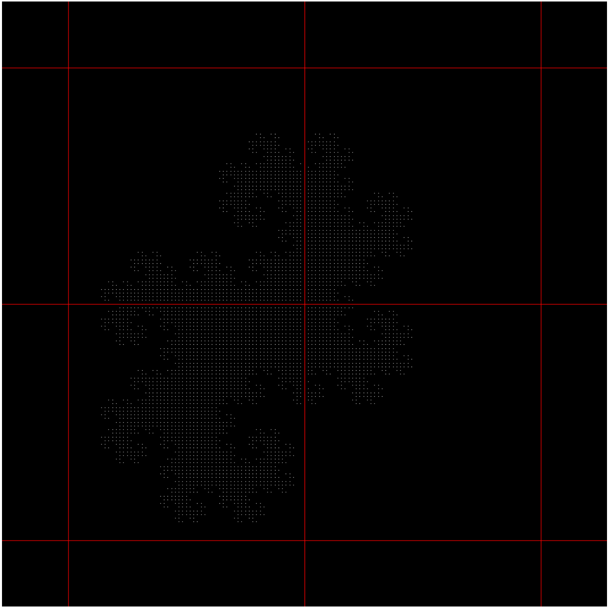
Expansions of length 8



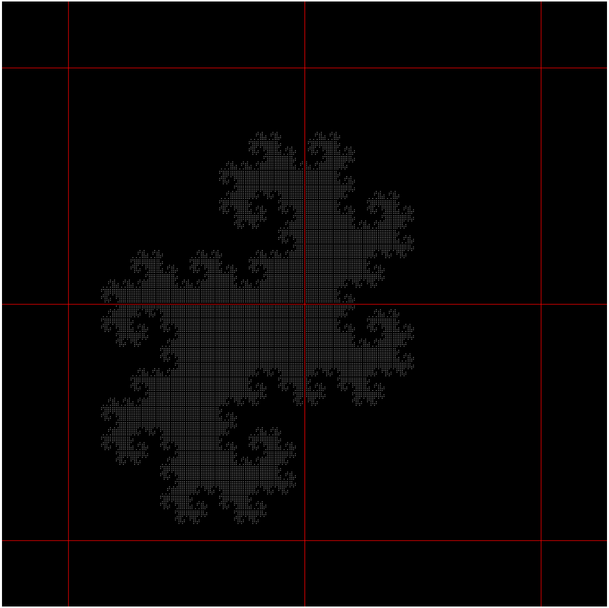
Expansions of length 10



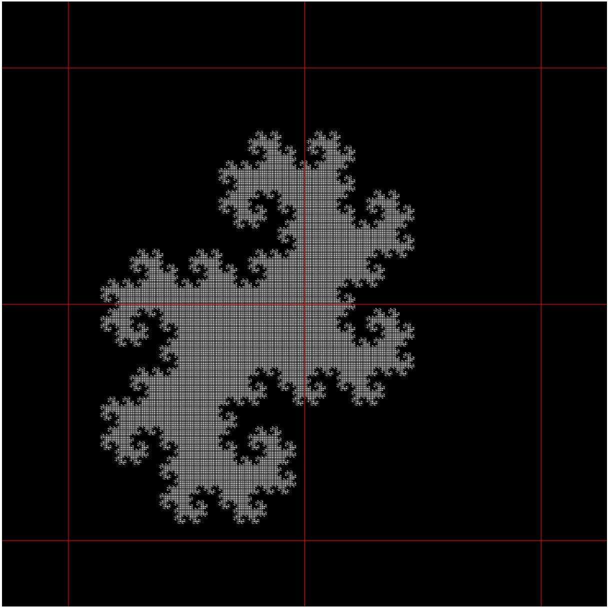
Expansions of length 12



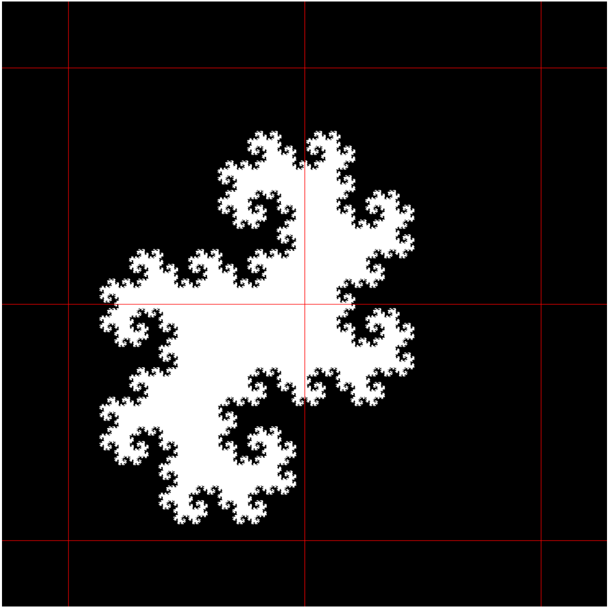
Expansions of length 14



Expansions of length 16



Expansions of length 18



Avižienis numeration

Fix a base $b \geqslant 3$ and a set of digits $A = \{-a, \ldots, -1, 0, 1, \ldots, a\}$ where $b/2 < a < b$.
For instance $b = 3$ and $A = \{-2, -1, 0, 1, 2\} = \{\overline{2}, \overline{1}, 0, 1, 2\}$

$$\widehat{d_n \cdots d_0} = \sum_{k=0}^{n} d_k b^k \quad \text{where} \quad d_k \in A$$

Base 3 expansion	Decimal value
102	11
$11\overline{1}$	11
$2\overline{2}\overline{1}$	11
$1\overline{1}\overline{2}\overline{1}$	11
$1\overline{2}02$	11
$1\overline{2}1\overline{1}$	11

Properties of Avižienis numeration system

- 0 has exactly one expansion because $a < b$,
- non-zero integers have several expansions because $b/2 < a$,
- non-zero integers have infinitely many expansions if $a = b - 1$,
- taking the opposite is easy,
- addition can be performed in parallel in constant time.

Addition in Avižienis numeration system

Suppose

$$x = \sum_{k=0}^{n} x_k b^k \qquad\qquad y = \sum_{k=0}^{n} y_k b^k$$

$$z = x + y = \sum_{k=0}^{n} z_k b^k$$

where

- For each k in parallel do
  1. $z_k := x_k + y_k$
  2. if $-a < z_k < a$ then $c_{k+1} := 0$
  3. if $z_k \geqslant a$ then $z_k := z_k - b$, $c_{k+1} := 1$
  4. if $z_k \leqslant -a$ then $z_k := z_k + b$, $c_{k+1} := -1$
  5. $z_k := z_k + c_k$